ON THE CAUCHY PROBLEM FOR THE LAMÉ SYSTEM

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Abstract. Solvability condition and Carleman formula for the Cauchy problem for the Lamé type system in $\mathbb{R}^n$ $(n \geq 2)$ are obtained. In particular, given displacement and stress on an open connected part of boundary of domain of special type, formula for reconstruction of solutions of the Lamé system in the domain is derived.

Erlösungsbedingungen und Carleman-Formel für das Cauchy-Problem für Systeme des Lamé-Typs im $\mathbb{R}^n$ $(n \geq 2)$ werden gegeben. Insbesondere wird für bestimmte Gebiete eine Formel zur Rekonstruktion von Lösungen des Lamé-Systems bei gegebener Translation und Verformung einer offenen, zusammenhängenden Teilmenge des Gebietsrandes entwickelt.

Получены условия разрешимости и формула Карлемана задачи Коши для системы типа ЛамЁ в $\mathbb{R}^n$ $(n \geq 2)$. В частности, построена формула для восстановления решений системы ЛамЁ в области специального вида, по заданным смещению и напряжению на открытом связном подмножестве границы области.

Introduction

Let G be a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ and S be a closed smooth hypersurface dividing it into 2 connected components: G^+ and $G^- = D$, and oriented as the boundary of G^-. In this paper we will consider the Cauchy problem for the system:

$$\mathcal{L}u(x) = f(x) \tag{1}$$

where $f(x) = (f_1(x), ..., f_n(x))^T$ is given, $u(x) = (u_1(x), ..., u_n(x))^T$ is unknown vector,

$$\mathcal{L} = \mu\, \Delta + (\lambda + \mu)\, \nabla\, div,$$

λ, μ are constants $(\mu \neq 0,\ \lambda \neq -2\mu)$ and, as usual, Δ is the Laplace operator, ∇ is the gradient operator and div is the divergence operator in $\mathbb{R}^n$.

In Elasticity Theory $(n = 2, 3)$, with *Lamé constants* λ, μ, this system is known as the *Lamé system*, f is the density vector of outer forces, and u is displacement.

More exactly, denoting by $\nu_j(x)$ the j-th component of the unit outward normal vector $\nu(x)$ to ∂D at the point x, by $\frac{\partial}{\partial \nu}$ the normal derivative with respect to ∂D and by T the *stress operator*, i.e the matrix $T(x, D) = (T_{ij}(x, D))_{i,j=1,2,...,n}$ with components

$$T_{ij}(x, D) = \mu\, \delta_{ij}\, \frac{\partial}{\partial \nu} + \lambda\, \nu_i(x)\, \frac{\partial}{\partial x_j} + \mu\, \nu_j(x)\, \frac{\partial}{\partial x_i}\ \ (i, j = 1, ..., n),$$

we consider the following problem.

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Problem 1. *Let* $u^0(x) = (u^0_1(x), ..., u^0_n(x))^T \in [C^1(S)]^n$, $u^1(x) = (u^1_1(x), ..., u^1_n(x))^T \in [C(S)]^n$, $f(x) \in [C(D)]^n$ *be given vector-functions. It requires to find (if possible) a vector-function* $u(x) \in [C^1(D \cup S) \cap C^2(D)]^n$ *such that*

$$\begin{cases} \mathcal{L}u = f \text{ in } D, \\ u_{|S} = u^0, \\ (Tu)_{|S} = u^1. \end{cases}$$

Here $[C^s(\Omega)]^n$ $(s = 0, 1, ...)$ stands for the vector space of all n-vector valued functions whose components are s times continuously differentiable on a set $\Omega \subset \mathbb{R}^n$.

It is known that the Lamé type system is elliptic and Problem 1 has no more than one solution (see, for example, [1], Theorem 2.8). However it is ill-posed, i.e. 1) not for any data there exists a solution; 2) solutions do not depend continuously on the Cauchy data on S (see Example 1.2 below). Therefore, solvability conditions can not be described in terms of continuous linear functionals (see [2]).

Of course, if the "piece" S and all the data are real analytic, there exist a solution of the Lamé system in a neighbourhood of S. However we are interested in global solvability.

In the paper [3], using ideas of [4] and [5], the regularization aspect of the Cauchy problem for the Lamé system in $\mathbb{R}^3$ was studied.

In [6], [7] and [1] special integral representation for solutions of elliptic systems were used to study the ill-posed Cauchy problem for elliptic systems. In this paper we will apply the integral representation's method to obtain solvability conditions and a formula for solutions of the problem.

In §1, using the Somigliana formula for solutions of the Lamé type system $\mathcal{L}$, we state the solvability criterion.

In §2, we present Carleman's formula for the determination of a solution of the Lamé type system in D, given its Cauchy data on S, in the case where G is a ball in $\mathbb{R}^n$ with centre at $0 \in G^+$. This formula involves the sum of a series whose terms are n-vectors H_ν $(\nu = 1, 2, ...)$ of homogeneous polynomials of degree ν (satisfying $\mathcal{L}H_\nu = 0$ in $\mathbb{R}^n$), while solvability of Problem 1 is equivalent to the convergence of the series.

§1. Criterion for solvability of Problem 1

As we already noted in introduction, the Lamé type system (1) is elliptic. Hence, solutions of the homogeneous Lamé type system $\mathcal{L}$ in the domain D (even generalized ones) are real analytic there. Let us first obtain a solvability criterion for the homogeneous case.

Problem 1.1. *Let* $w^0(x) = (w^0_1(x), ..., w^0_n(x))^T \in [C^1(S)]^n$, $w^1(x) = (w^1_1(x), ..., w^1_n(x))^T \in [C(S)]^n$ *be given vector-functions. It requires to find (if possible) a vector-function* $w(x) \in [C^1(D \cup S)]^n$ *such that*

$$\begin{cases} \mathcal{L}w = 0 \text{ in } D, \\ w_{|S} = w^0, \\ (Tw)_{|S} = w^1. \end{cases}$$

It is known that Problem 1.1 has no more than one solution (see, for example, [1] Theorem 2.8). That the Cauchy Problem 1.1 for the Lamé type system is ill-posed, is demonstrated by a simple example similar to that of Hadamard [8].

Example 1.2. Let S be a piece of the plane $\{x_3 = 0\}$ in $\mathbb{R}^3$ and

$$u^{(\nu)}(x) = \nabla\left(\frac{\sin\nu x_1\,\sin\nu x_2\,\sinh\sqrt{2}\,\nu x_3}{\nu^3}\right),\quad \nu = 1, 2, \ldots .$$

Each $u^{(\nu)}$ is easily verified to be a solution of the Lamé system on $\mathbb{R}^3$. Moreover,

$$|u^{(\nu)}| \le \frac{1}{\nu^2}\ \mathit{for}\ x_3 = 0,$$

$$|Tu_{(\nu)}| \le \frac{4\mu}{\nu}\ \mathit{for}\ x_3 = 0.$$

However, at each point $x = (x_1, x_2, x_3)$ with $x_1 \neq 0$, $x_2 \neq 0$ and $x_3 > 0$, we have

$$\lim_{\nu\to\infty} u_{(\nu)}(x) = \infty. \qquad \square$$

To study (the Cauchy) Problem 1.1 we will use a suitable fundamental solution of the homogeneous Lamé type system $\mathcal{L}$.

We denote by σ_n the area of the unit sphere in $\mathbb{R}^n$, and by $g(y)$ the standard (bilateral) fundamental solution of the Laplace operator in $\mathbb{R}^n$:

$$g(x) = \begin{cases} \frac{1}{(2-n)\sigma_n}\frac{1}{|x|^{n-2}}, & n > 2,\\ \frac{1}{2\pi} ln|x|, & n = 2.\end{cases}$$

Lemma 1.3. *The matrix* $\Phi(x) = (\Phi_{ij}(x))_{i,j=1,2,\ldots,n}$ *with components*

$$\Phi_{ij}(x) = \frac{1}{2\mu(\lambda+2\mu)}\left(\delta_{ij}\,(\lambda+3\mu)g(x) - (\lambda+\mu)\,x_j\,\frac{\partial}{\partial x_i}g(x)\right)\quad (i,j = 1, 2, ..., n),$$

where δ_{ij} *is the Kronecker delta, is a fundamental solution of convolution type for the homogeneous Lamé type system* $\mathcal{L}$.

Proof. It is proved by direct calculations. $\square$

The matrix Φ is called the *Kelvin-Somigliana matrix* for $n = 3$ (see, for example, [9]).

Lemma 1.4 (Somigliana formula). *Let* D *be a domain with a piecewise smooth boundary* ∂D. *Then for any solution* $u \in [C^1(\overline{D})]^n$ *of the homogeneous Lamé type system in* D, *it follows that*

$$\int_{\partial D}\left((T(y,D)\Phi(x-y))^T u(y) - \Phi(x-y)\,T(y,D)u(y)\right) ds(y) = \begin{cases} u(x), & x\in D,\\ 0, & x\in\mathbb{R}^n\backslash\overline{D}.\end{cases} \tag{1.1}$$

Proof. It follows from Lemma 1.3 and the Stokes' formula (see, for example, [9], for $n = 3$). $\square$

From now on we will assume that D is a domain with a piecewise smooth boundary ∂D, and that the vector-functions w^0, w^1 are summable on S. Then the following integral is well defined:

$$\mathcal{F}(x) = \int_S \left((T(y,D)\Phi(x-y))^T w^0(y) - \Phi(x-y)\,w^1(y)\right) ds(y)\ \ (x \in G\backslash S). \tag{1.2}$$

It is clear that $\mathcal{F}$ is a solution of the homogeneous Lamé type system everywhere outside of S; let $\mathcal{F}^{\pm} = \mathcal{F}_{|G^{\pm}}$.

Lemma 1.5. *Let $S \in C^2$, $w^0 \in [C^1(S)]^n$ and $w^1 \in [C(S)]^n$ be summable functions on S. Then the vector-function $\mathcal{F}^+$ continuously extends to $G^+ \cup S$ together with its first derivatives if and only if the function $\mathcal{F}^-$ continuously extends to $G^- \cup S$ together with its first derivatives.*

Proof. We will use the fact that there exist a smooth function $\hat{w}$ given in a neighbourhood of S in G such that $\hat{w}_{|S} = w^0$, $(T\hat{w})_{|S} = w^1$ (see [10], Lemma 29.5).

If $x^0 \in S$, $\nu(x^0)$ is the unit normal vector to S at the point x^0 and $|\alpha| \leq 1$ then (see [10], Lemma 29.5)

$$\lim_{\varepsilon \to 0} (\partial^\alpha \mathcal{F})(x^0 - \varepsilon\nu(x^0)) - \partial^\alpha \mathcal{F}(x^0 + \varepsilon\nu(x^0))) = \partial^\alpha \hat{w}(x^0), \tag{1.3}$$

where the limit is uniform on compact subsets in S.

Let, for instance, $\mathcal{F}^-$ continuously extends to $G^- \cup S$ together with its first derivatives. We fix a multi-index $|\alpha| \leq 1$. Then

$$\lim_{\varepsilon \to 0} \partial^\alpha \mathcal{F}(x^0 + \varepsilon\nu(x^0)) = \partial^\alpha \mathcal{F}^-(x^0) - \partial^\alpha \hat{w}(x^0).$$

Let us define $\mathcal{F}^+$ in the following way:

$$\partial^\alpha \mathcal{F}^+(x) = \begin{cases} \partial^\alpha \mathcal{F}^+(x), \ x \in G^+, \\ \partial^\alpha \mathcal{F}^-(x) - \partial^\alpha \hat{w}(x), \ x \in S. \end{cases}$$

Let us show that $\partial^\alpha \mathcal{F}^+$ is continuous in $G^- \cup S$. We fix a point $x^0 \in S$ and $E > 0$. Because $\partial^\alpha \mathcal{F}^+$ is continuous on S, there is $\delta_0 > 0$ such that, for $x^1 \in S$ with $|x^1 - x^0| < \delta_0$, we have

$$|\partial^\alpha \mathcal{F}^+(x^1) - \partial^\alpha \mathcal{F}^+(x^0)| < E/2.$$

Decreasing δ_0 (if it is necessary) we can consider $K = \overline{B(x^0, \delta_0)} \cap S$ as a compact subset of S.

Since the hypersurface $S \in C^2$, we can choose $0 < \delta < \delta_0$ in such a way that every point $x \in G^+ \cap B(x^0, \delta)$ is represented in the form $x = x^1 + \varepsilon\nu(x^1)$ where $x^1 \in S$ and $\varepsilon = dist(x, S)$. Then $\varepsilon < \delta$ and $|x^0 - x^1| \leq |x^0 - x| + |x - x^1|$, i.e. $x^1 \in K$.

Using the fact that the limit in (1.3) is uniform on compact subsets of S and decreasing δ (if it is necessary) we obtain that, for $x^1 \in K$, $0 < \varepsilon < \delta$ the following inequality holds:

$$|\partial^\alpha \mathcal{F}^+(x^1 + \varepsilon\nu(x^1)) - \partial^\alpha \mathcal{F}^+(x^1)| < E/2.$$

Let now $x \in D^+ \cap B(x^0, \delta)$. Then, for some $x^1 \in K$ and $0 < \varepsilon < \delta$ we have $x = x^1 + \varepsilon\nu(x^1)$. Hence

$$|\partial^\alpha \mathcal{F}^+(x^0) - \partial^\alpha \mathcal{F}^+(x)| \leq |\partial^\alpha \mathcal{F}^+(x^0) - \partial^\alpha \mathcal{F}^+(x^1)| +$$

$$+ |\partial^\alpha \mathcal{F}^+(x^1 + \varepsilon\nu(x^1)) - \partial^\alpha \mathcal{F}^+(x^1)| < E.$$

Therefore $\mathcal{F}^+$ continuously extends to $G^+ \cup S$ together with its first derivatives, if $\mathcal{F}^-$ continuously extends to $G^- \cup S$ together with its first derivatives. The proof is complete. $\square$

Theorem 1.6. *Let $S \in C^2$, $w^0 \in [C^1(S)]^n$ and $w_1 \in [C(S)]^n$ be summable vector-functions on S. Then, for Problem 1.1 to be solvable, it is necessary and sufficient that the integral $\mathcal{F}^+$ extends, as a solution of the homogeneous Lamé type system, from G^+ to the domain G.*

Proof. Necessity. Suppose that there exists a function w that solves Problem 1.1. Define in G the function

$$\mathfrak{F}(x) = \begin{cases} \mathcal{F}^+(x), & x \in G^+ \\ \mathcal{F}^- - w(x), & x \in G^-. \end{cases} \tag{1.4}$$

For any subdomain $S_1 \subset S$ there is some domain $D_1 \Subset G$ in D with a piecewise smooth boundary such that $S_1 \subset \partial D_1$. Clearly, $\in C^1(\overline{D}_1)$ is a solution of the homogeneous Lamé type system in D_1 and so, by formula (1.1)

$$\int_{\partial D_1} \left((T(y,D)\Phi(x-y))^t \, w(y) - \Phi(x-y)\, T(y,D) w(y) \right) ds(y) = w(x), \;\; (x \in D_1)$$

Hence we have, in D_1,

$$\mathfrak{F}(x) = \mathcal{F}^-(x) - w(x) = \int_{S \setminus S_1} \left((T(y,D)\Phi(x-y))^t \, w^0(y) - \Phi(x-y)\, w^1(y) \right) ds(y) +$$

$$+ \int_{\partial D_1 \setminus S_1} \left((T(y,D)\Phi(x-y))^t \, w(y) - \Phi(x-y)\, T(y,D) w(y) \right) ds(y) \;\; (x \in D_1). \tag{1.5}$$

The terms in the right hand side of (1.5) are solutions of the homogeneous Lamé type system in a neighbourhood of S_1, and therefore, since S_1 is arbitrary, $\mathcal{F}^-$ extends smoothly to $G^- \cup S$.

Further, it follows from Lemma 1.5 that $\mathcal{F}^+$ also extends smoothly to $D^+ \cup S$. Therefore, the restriction $\mathfrak{F}^{\pm}$ to $G^{\pm}$ of $\mathfrak{F}$ extends smoothly to $G^{\pm} \cup S$. In addition, by (1.3), if $x^0 \in S$, then

$$\lim_{\varepsilon \to 0} \left(\mathfrak{F}^-(x^0 - \varepsilon\nu(x^0)) - \mathfrak{F}^+(x^0 + \varepsilon\nu(x^0)) \right) = 0,$$

$$\lim_{\varepsilon \to 0} \left(\frac{\partial \mathfrak{F}^-}{\partial n}(x^0 - \varepsilon\nu(x^0)) - \frac{\partial \mathfrak{F}^+}{\partial n}(x^0 + \varepsilon\nu(x^0)) \right) = 0.$$

Thus, we conclude that $\mathfrak{F}$ can be extended smoothly to the whole domain G by defining $\mathfrak{F} = \mathcal{F}^- - w$ on S.

By Theorem 3.2 of [10], it follows that a function $\mathfrak{F}$ that is smooth in G and a solution of the homogeneous Lamé type system in G^- and in G^+ is also a solution of the homogeneous Lamé type system in G. By (1.4), $\mathfrak{F}$ is the desired extension of $\mathcal{F}^+$ to G.

Sufficiency. Let $\mathcal{F}^+$ be extendable to a solution of the homogeneous Lamé type system in G, call it $\mathfrak{F}$. Then, by Lemma 1.5, $\mathcal{F}^-$ extends smoothly to $G^- \cup S$. Define $w(x) = \mathcal{F}^- - \mathfrak{F}(x)$ $(x \in D)$. Using formula (1.3) as in the proof of the necessity, we see that the restriction to S of w and Tw equal w_0 and w_1, respectively. □

A similar statement was obtained for the Cauchy-Riemann system in [6], for the Laplace equation in [7], and for rather general elliptic system in [1].

Let now $f \in [C^{0,\gamma}(\overline{D})]^n$ with a Hölder degree $\gamma > 0$. Then, since Φ is a bilateral fundamental solution, the vector-function

$$(Vf)(x) = \int_D \Phi(x-y)f(y)dy \ (x \in \mathbb{R}^n)$$

is a solution of the Lamé type system (1) in D. Moreover, if and $S \in C^2$, $(Vf) \in [C^{2,\gamma}(D \cup S)]^n$. Hence, if $f \in [C^{0,\gamma}(\overline{D})]^n$ and $S \in C^2$, Problem 1 is equivalent to Problem 1.1 with $w^0 = u^0 - (Vf)_{|S}$, $w^1 = u^1 - (TVf)_{|S}$.

Assume that the vector-functions u^0, u^1 are summable on S. Then the vector functions $w^0 = u_0 - (V(f))_{|S}$ and $w^1 = u_1 - (TV(f))_{|S}$ are summable too and the integral $\mathcal{F}$ is well defined. Therefore we obtain

Corollary 1.7. *Let $S \in C^2$, $f \in [C^{0,\gamma}(\overline{D})]^n$ ($\gamma > 0$), and let $u^0 \in [C^1(S)]^n$ and $u_1 \in [C(S)]^n$ be summable vector-functions on S. Then, for Problem 1 to be solvable, it is necessary and sufficient that the integral $\mathcal{F}^+$, with $w^0 = u^0 - (V(f))_{|S}$, $w^1 = u^1 - (TV(f))_{|S}$, extends, as a real analytic vector-function, from G^+ to the domain G.*

Proof. Follows from Theorem 1.6, Uniqueness Theorem for real analytic functions and real analyticity of solutions of the homogeneous Lamé type system. □

Obviously, if $w^0 = u^0 - (Vf)_{|S}$, $w^1 = u^1 - (TVf)_{|S}$ and Problem 1 (or Problem 1.1) is solvable with solution u (or w, respectively), then

$$u(x) = V(f)(x) + w(x) \ (x \in D).$$

Using (1.4) we can obtain a formula for reconstruction of a solution of the Lamé type system (1), given its Cauchy data on S.

Corollary 1.8. *Let $S \in C^2$, $f \in [C^{0,\gamma}(\overline{D})]^n$ ($\gamma > 0$). Then, for any solution $u \in [C^1(S \cup D)]^n$ of the Lamé type system (1) such that $u_{|S}$ and $(Tu)_{|S}$ are summable on S, the following formula holds:*

$$u(x) = \int_D \Phi(x-y)f(y)dy \ + \mathcal{F}^-(x) - \mathfrak{F}(x) \ (x \in D),$$

where $\mathcal{F}$ defined by (1.2) with $w^0 = u^0 - (Vf)_{|S}$, $w^1 = u^1 - (TVf)_{|S}$, and $\mathfrak{F}$ is its analytic extension from G^+ to G.

Proof. Follows from formula (1.4), Corollary 1.7, and the fact that solution of the Cauchy Problem 1 is unique. □

Because $\mathcal{F}$ is real analytic everywhere outside of S, Corollary 1.8 implies that the behaviour of the solution u of the Cauchy Problem 1 near $\partial D \backslash S$ is completely determined by the behaviour of the real analytic extension $\mathfrak{F}$ of the integral $\mathcal{F}^+$ near $\partial D \backslash S$, at least if $\partial D \in C^2$.

§2. The case where G is a ball.

Let $G = B_R$ be the ball with centre at zero and radius $0 < R < \infty$, and S be a closed smooth hypersurface dividing it into 2 connected components (G^+ and G^-) in such a way that $0 \in G^+$, and oriented as the boundary of G^-.

Let $\{h_\nu^{(i)}\}$ be a set of homogeneous harmonic polynomials which form a complete orthonormal system in $L^2(\partial B_1)$ (*spherical harmonics*), where ν is the degree of homogeneity, and i is an index labelling the polynomials of degree ν belonging to the basis . The size of the index set for i as a function of ν is known, namely, $1 \le i \le J(\nu)$ where $J(\nu) = \frac{(n+2\nu-2)(n+\nu-3)!}{\nu!(n-2)!}$ for $n > 2$ and $\nu \ge 0$. If $n = 2$ then, obviously, $J(0) = 1$, $J(\nu) = 2$ for $\nu \ge 1$ (see [11], p. 453).

Lemma 2.1. *The fundamental solution of the Laplace operator can be expanded as follows:*

$$g(x-y) = g(y) - \sum_{\nu=1}^{\infty} \sum_{i=1}^{J(\nu)} \frac{h_\nu^{(i)}(x)}{(n+2\nu-2)} \frac{\overline{h_\nu^{(i)}(y)}}{|y|^{n+2\nu-2}}. \tag{2.1}$$

where the series converges absolutely together with all the derivatives, uniformly on compact subsets of the cone $\mathcal{K} = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \ : \ |y| > |x|\}$.

Proof. It was proved in [7], Lemma 3.2, that the series converges together with all the derivatives uniformly on compact subsets of the cone $\mathcal{K} = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \ : \ |y| > |x|\}$. Hence the statement of the lemma follows, for example, from the following estimate of a harmonic homogeneous polinomial h_ν of degree ν on the unit sphere (see [11]):

$$\max_{|y|=1} |h_\nu| \le const(n) \nu^{n/2-1} \|h_\nu\|_{L^2(\partial B_1)}. \qquad \square \tag{2.2}$$

Using this lemma we obtain the following decomposition for the fundamental solution Φ of the homogeneous Lamé type system.

Lemma 2.2. *The fundamental solution $\Phi(x-y)$ of the homogeneous Lamé type system $\mathcal{L}$ can be expanded as follows:*

$$\Phi(x-y) = \sum_{\nu=0}^{\infty} \Phi^{(\nu)}(x,y)$$

where the series converges absolutely together with all the derivatives uniformly on compact subsets of the cone $\mathcal{K} = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \ : \ |y| > |x|\}$ and $\Phi^{(\nu)}(x,y)$ ($\nu \ge 0$) are matrix with compoments $\Phi_{kl}^{(\nu)}(x,y)$ ($k,l = 1,2,...,n$):

$$\Phi_{kl}^{(0)}(x,y) = g(y) \frac{(\lambda+3\mu)\delta_{kl}}{2\mu(\lambda+2\mu)} - \sum_{i=1}^{n} \frac{\lambda+\mu}{2\mu(\lambda+2\mu)} \frac{y_l \, \overline{h_1^{(i)}(y)}}{n \, |y|^n} \frac{\partial h_1^{(i)}(x)}{\partial x_k},$$

$$\Phi_{kl}^{(\nu)}(x,y) = - \sum_{i=1}^{J(\nu)} \frac{\overline{h_\nu^{(i)}(y)}}{|y|^{n+2\nu-2}} \left(\frac{\lambda+3\mu}{2\mu(\lambda+2\mu)} \frac{\delta_{kl} \, h_\nu^{(i)}(x)}{(n+2\nu-2)} - \right.$$

$$\left. - \frac{\lambda+\mu}{2\mu(\lambda+2\mu)} \frac{\partial h_\nu^{(i)}(x)}{\partial x_k} \frac{x_l}{(n+2\nu-2)} \right) -$$

$$- \sum_{i=1}^{J(\nu+1)} \frac{\lambda+\mu}{2\mu(\lambda+2\mu)} \frac{\overline{h_{\nu+1}^{(i)}(y)}}{|y|^{n+2\nu}} \frac{y_l}{(n+2\nu)} \frac{\partial h_{\nu+1}^{(i)}(x)}{\partial x_k} \ (\nu \ge 1). \tag{2.3}$$

Lemma 2.3. *For $\nu = 1, 2, ...$ and $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n \setminus \{0\}$ we have*

$$\mathcal{L}_x \Phi^{(\nu)}_{kl}(x,y) = 0,$$

$$\Delta^2_y \Phi^{(\nu)}_{kl}(x,y) = 0.$$

Proof. Due to the harmonicity of the polynomials $h^{(i)}_\nu$, it follows from Lemma 1.3 that the matrix, whose components are formed by the first sum in the right hand side of (2.3), is a solution of the homogeneous Lamé type system.

On the other hand, the matrix, whose components are formed by the second sum in the right hand side of (2.3), is equal to

$$\sum_{i=1}^{J(\nu+1)} \frac{\lambda+\mu}{2\mu(\lambda+2\mu)} \frac{\overline{h^{(i)}_{\nu+1}(y)}}{|y|^{n+2\nu}} \frac{\nabla_x h^{(i)}_{\nu+1}(x) y^T}{(n+2\nu)}.$$

Therefore, because of the harmonicity of the polynomials $h^{(i)}_\nu$, it is a solution of the homogeneous Lamé type system too.

The second equality is obvious. □

Let us obtain now a decomposition of the vector-function $\mathcal{F}$ in a neighbourhood of origin.

Lemma 2.4. *Let $0 < \rho < dist(0, S)$ be fixed, so that the ball $B_\rho \Subset G^+$. Then*

$$\mathcal{F}^+(x) = \sum_{\nu=0}^{\infty} H_\nu(x) \ (x \in B_\rho), \tag{2.4}$$

where the series converges absolutely together with all the derivatives uniformly on compact subsets of the ball B_ρ and H_ν are n- vectors of homogeneous polynomials of degree ν, satisfying $\mathcal{L}\, H_\nu = 0$ in $\mathbb{R}^n$:

$$H_\nu(x) = \int_S \left((T(y,D)\Phi^{(\nu)}(x,y))^T\, w^0(y) - \Phi^{(\nu)}(x,y)\, w^1(y) \right) ds(y).$$

Proof. Since $0 \notin S$,

$$\max_{x \in B_\rho, y \in \overline{S}} \frac{|x|}{|y|} \le q < 1.$$

Then, using estimate (2.2), one easily obtains that

$$|H_\nu(x)| \le C\ q^{2\nu}(\nu+1)^{n-2} \ (x \in B_\rho,\ \nu \ge 0) \tag{2.5}$$

with a constant $C > 0$ which depends on w^0, w^1 and does not depend on ν and x. Estimate (2.5) implies that the series $\sum_{\nu=0}^{\infty} H_\nu(x)$ converges absolutely together with all the derivatives uniformly on compact subsets of the ball B_ρ. Now, using formula (1.2), Lemmata 2.2, and 2.3, we conclude that the statement of the lemma holds. □

Proposition 2.5. *Let $S \in C^2$, $w^0 \in [C^1(S)]^n$ and $w_1 \in [C(S)]^n$ be summable vector-functions on S. Then, for Problem 1.1 to be solvable, it is necessary and sufficient that the series $\sum_{\nu=0}^{\infty} H_\nu(x)$ converges absolutely together with all the derivatives uniformly on compact subsets of the ball B_R.*

Proof. Necessity. Let Problem 1.1 be solvable. Then Theorem 1.6 implies that the integral $\mathcal{F}^+$ on the domain G^+ extends to a solution $\mathfrak{F}$ of the homogeneous Lamé type system in B_R.

We fix $0 < r_0 < R$. It is clear that $\mathfrak{F} \in [C^1(\overline{B}_{r_0})]^n$. Hence, it represents in the ball B_{r_0} in the following way (see Lemma 1.4):

$$\mathfrak{F}(x) = \int_{\partial B_{r_0}} \left((T(y,D)\Phi(x-y))^T \mathfrak{F}(y) - \Phi(x-y)\, T(y,D)\mathfrak{F}(y)\right) ds(y).$$

Substituting instead of Φ its decomposition obtained in Lemma 2.2 and arguing in the same way as in Lemma 2.4, we obtain that, for $x \in B_r$ $(0 < r < r_0)$,

$$\mathfrak{F}(x) = \sum_{\nu=0}^{\infty} \widetilde{H}_\nu(x) \ (x \in B_r), \tag{2.6}$$

where the series converges absolutely together with all the derivatives uniformly on compact subsets of the ball B_r and $\widetilde{H}_\nu$ are n-vectors of homogeneous polinomials of degree ν and solutions of the homogeneous Lamé type system $\mathcal{L}$ in $\mathbb{R}^n$:

$$\widetilde{H}_\nu(x) = \int_{\partial B_{r_0}} \left((T(y,D)\Phi^{(\nu)}(x,y))^T \mathfrak{F}(y) - \Phi^{(\nu)}(x,y)\, T(y,D)\mathfrak{F}(y)\right) ds(y).$$

Comparing (2.4) and (2.6) we find that

$$D^\alpha H_\nu = D^\alpha \widetilde{H}_\nu = D^\alpha(\mathcal{F}^+)(0) \ \ (|\alpha| = \nu,\ \nu = 0, 1, ...).$$

Because H_ν, $\widetilde{H}_\nu$ are homogeneous , we conclude that, for $x \in \mathbb{R}^n$,

$$H_\nu(x) = \widetilde{H}_\nu(x) \ \ (\nu = 0, 1, ...).$$

Therefore the series $\sum_{\nu=0}^{\infty} H_\nu(x)$ converges absolutely together with all the derivatives uniformly on compact subsets of the ball B_r.

Since $0 < r_0 < R$ is arbitrary then the series $\sum_{\nu=0}^{\infty} H_\nu(x)$ converges absolutely together with all the derivatives uniformly on compact subsets of the ball B_R, which was to be proved.

Sufficiency. Let the series $\mathfrak{F}(x) = \sum_{\nu=0}^{\infty} H_\nu(x)$ converge absolutely together with all the derivatives uniformly on compact subsets of the ball B_R. Since the polinomials H_ν are solutions of the homogeneous Lamé type system in $\mathbb{R}^n$, by Stiltjes-Vitali theorem we conclude that $\mathfrak{F}$ satisfies $\mathcal{L}\mathfrak{F} = 0$ in B_R.

It is easy to see from Lemma 2.4, that in a neighbourhood of the origin the vector-function $\mathfrak{F}$ coincides with $\mathcal{F}^+$. Now, using Theorem 1.6 and the Uniqueness theorem for real analytic functions, we conclude that Problem 1.1 is solvable. This proves the proposition. □

Proposition 2.5 can be used to prove Carleman's formula for determination of a solution u of the Lamé type system (1) in B_R by its Cauchy data on S.

For each number $N = 1, 2...$ we consider the kernel $\mathfrak{C}^{(N)}(x,y)$ defined, for all $y \neq 0$ off the diagonal $\{x = y\}$, by the equality

$$\mathfrak{C}^{(N)}(x,y) = \Phi(x-y) - \sum_{\nu=0}^{N} \Phi^{(\nu)}(x,y).$$

Proposition 2.6. *For any number $N = 1, 2, \ldots$, the kernel $\mathfrak{C}^{(N)}$ satisfies the equations $\mathcal{L}_x \mathfrak{C}^{(N)}(x,y) = 0$, $\Delta_y^2 \mathfrak{C}^{(N)}(x,y) = 0$ for all $y \neq 0$ off the diagonal $\{x = y\}$.*

Proof. Follows from the properties of the $\Phi(x-y)$ and Lemma 2.3. □

The following formula produces rather explicitely a way to obtain a solution of the Lamé type system (1) by successive approximations.

Theorem 2.7 (Carleman's formula). *Let $S \in C^2$, $f \in [C^{0,\gamma}(\overline{D})]^n$ ($\gamma > 0$). Then, for any solution $u \in C^1(D \cup S)$ of the Lamé type system (1) such that $u_{|S}$ and $(Tu)_{|S}$ are summable on S, the following formula holds:*

$$u(x) = \int_D \Phi(x-y) f(y) dy +$$

$$+ \lim_{N\to\infty} \int_S \Big((T(y,D)\mathfrak{C}^{(N)}(x,y))^T w^0(y) - \mathfrak{C}^{(N)}(x,y)\, w^1(y) \Big) ds(y) \qquad (2.7)$$

where $w^0 = (u - V(f))_{|S}$, $w^1 = (Tu - TV(f))_{|S}$.

Proof. Since $w = u - V(f)$ is a solution of Problem 1.1 with the Cauchy data $w^0 = (u - V(f))_{|S}$, $w^1 = (Tu - TV(f))_{|S}$ on S, it follows from Theorem 1.6 that $\mathcal{F}^+$ extends to the ball B_R, as a solution the homogeneous Lamé type system, say $\mathfrak{F}$. It is evident from Theorem 1.6 that the function $\hat{w} = \mathcal{F}^- - \mathfrak{F}$ is also a solution of Problem 1.1. It is readily seen that in that case $\hat{w}$ coincides with w in D.

Proving Proposition 2.5 we have seen that, in the ball B_R, we have the decomposition $\mathfrak{F}(x) = \sum_{\nu=0}^{\infty} H_\nu(x)$, where the series converges absolutely and uniformly on compact subsets of B_R. Because any point $x \in D$ also belongs to some ball B_r, smaller than B_R, we conclude that

$$w(x) = \mathcal{F}^-(x) - \mathfrak{F}(x) = \mathcal{F}^-(x) - \sum_{\nu=0}^{\infty} H_\nu(x) =$$

$$= \int_S \left((T(y,D)\Phi(x-y))^T w^0(y) - \Phi(x-y)\, w^1(y) \right) ds(y) -$$

$$- \sum_{\nu=0}^{\infty} \int_S \Big((T(y,D)\Phi^{(\nu)}(x,y))^T w^0(y) - \Phi^{(\nu)}(x,y)\, w^1(y) \Big) ds(y).$$

Since the point zero is not on S, by assumption of the theorem, we have

$$w(x) = \lim_{N\to\infty} \int_S \Big((T(y,D)\mathfrak{C}^{(N)}(x,y))^T w^0(y) - \mathfrak{C}^{(N)}(x,y)\, w^1(y) \Big) ds(y) \ (x \in D).$$

Bacause $w = u - V(f)$, we conclude that (2.7) holds. □

Remark 2.8. As one can see from the proof of Theorem 2.7, the convergence of the limit in (2.7) is uniform on compact subsets of $D \cup S$ together with all its derivatives.

A Carleman formula for solutions of the Lamé system in $\mathbb{R}^3$ was established in [3] for specific choices of D, for example if it is bounded by part of the surface of a cone $\mathcal{K}$ and a smooth piece of S in the interior of $\mathcal{K}$, or if it is a relatively compact domain in $\mathbb{R}^3$ whose boundary consists of a piece of the plane $\{x_3 = 0\}$ and a smooth surface S lying in the half-space $\{x_3 > 0\}$.

If $\lambda = -\mu$, formula (2.7) is Carleman's formula for harmonic functions obtained in [7].

Example 2.9. If n=2 then, as a system of harmonic homogeneous polinomials $\{h_\nu^{(i)}\}$ we can take the system $\{\frac{1}{\sqrt{2\pi}}, \frac{z^\nu}{\sqrt{2\pi}}, \frac{\overline{z}^\nu}{\sqrt{2\pi}}\}$ where $z = x_1 + \sqrt{-1}x_2$, $(x_1, x_2) \in \mathbb{R}^2$.