æ **BASES WITH DOUBLE ORTHOGONALITY IN THE CAUCHY PROBLEM FOR SYSTEMS WITH INJECTIVE SYMBOLS**

A.A. Shlapunov, N.N. Tarkhanov[1]

Abstract. Principles for applications of double orthogonality bases in the Cauchy problem for systems with injective symbols are worked out. We obtain a solvability condition and a Carleman formula for the solution of the problem.

[Received 6 November 1991 – Revised 29 March 1993 and November 1993]

CONTENTS

[1]This research was partially supported by Alexander von Humboldt foundation.
Proc. London Math. Soc. (3), 71 (1995), 1-54

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX
Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

æ

INTRODUCTION

We shall be considering the Cauchy problem for solutions of a differential equation $Pf = 0$ where $P \in do_p(E \to F)$ is a differential operator with an injective symbol on an open set $X \subset \mathbb{R}^n$. Here $E = X \times \mathbb{C}^k$ and $F = X \times \mathbb{C}^l$ are (trivial) vector bundles over X whose sections of the class $\mathfrak{C}$ over an open set $\sigma \subset X$ are interpreted as columns of functions from $\mathfrak{C}(\sigma)$, that is, $\mathfrak{C}(E_{|\sigma}) = [\mathfrak{C}(\sigma)]^k$, and similarly for F, and the sign $do_p(E \to F)$ means the vector space of all the differential operators of type $(E \to F)$ and order $\leq p$.

In this way the differential operator P is given by an $(l \times k)$-matrix of scalar differential operators whose orders are less or equal than p on X, or $P(x, D) = \sum_{|\alpha| \leq p} P_\alpha(x) D^\alpha$ where $P_\alpha(x)$ are $(l \times k)$-matrices of (infinitely) differentiable functions on X. Then the injectivity of the symbol of the differential operator P means that $rank_{\mathbb{C}} \sigma(P)(x, \zeta) = k$ for all $(x, \zeta) \in X \times \mathbb{R}^n \setminus \{0\}$.

The most important class of operators with injective symbols is the class of elliptic differential operators corresponding to the case $l = k$. The model example of other types of systems is the Cauchy-Riemann system in the space $\mathbb{C}^n$, of dimension $n > 1$.

As in the last example, under sufficiently broad assumptions about the differential operator P, it is possible to include it in some elliptic complex of differential operators on X, say, $\{E^i, P^i\}$ where $E^i = X \times C^{k_i}$ are (trivial) vector bundles over X which are different from zero only for $0 \leq i \leq N$, and $P^i \in do_{p_i}(E^i \to E^{i+1})$ where $P^0 = P$ (see Samborskii [48]). We shall often use this identification, assuming that the conditions on P are fulfilled.

If the differential operator P has injective symbol then P is hypoelliptic; that is, for any distribution $f \in \mathcal{D}'(E)$ the singular supports of f and Pf $(\in \mathcal{D}'(F))$ coincide. In particular, for any open set $\sigma \subset X$ all generalized solutions $f \in \mathcal{D}'(E_{|\sigma})$ of the system $Pf = 0$ on σ are in fact (infinitely) differentiable.

Certainly, an open set is the natural domain of the system $Pf = 0$. However some problems require the consideration of solutions on sets $\sigma \subset X$ which are not open. Here we are interested not simply in restrictions of solutions to the given set, but also in the so-called local solutions of the system $Pf = 0$ on σ, that is, solutions of this system in a neighbourhood of σ. The space of local solutions of the system $Pf = 0$ on σ will be denoted by $S(\sigma)$.

We always suppose that P satisfies the so-called uniqueness condition of the Cauchy problem in the small on X:

$(U)_S$ *if for a domain* $O \subset X$ *we have* $f \in S(O)$*, and* $f = 0$ *on a non-empty open subset of* O *then* $f \equiv 0$ *in* O.

We suppose now that D is a relatively compact domain in X with a sufficiently smooth boundary, and that S is a set of positive ($(n-1)$-dimensional) measure on the boundary of D. The rough wording of the Cauchy problem for solutions of the system $Pf = 0$ in D with the data on S consists of the following.

PROBLEM 1. *Let* f_α ($|\alpha| \le p-1$) *be given sections of* E *over* S*. It is required to find a solution* $f \in S(D)$ *whose derivatives* $D^\alpha f$ *up to order* $(p-1)$ *have, in a suitable sense, limit values* $D^\alpha f_{|S}$ *on* S *such that* $D^\alpha f_{|S} = f_\alpha$ ($|\alpha \le p-1$).

Since the time of Hadamard, this problem has been known as the classic example of an ill-posed problem (see Hadamard [14], p.39). However, despite Hadamard's bold thinking, we often come across with these problems in applications of mathematics (see Hadamard [14], p.38). For example, the Cauchy problem for the Laplace equation naturally arises in problems of the interpretation of electrical prospecting data.

The Cauchy problem for the Laplace operator in various forms has been studied by Mergeljan [38], Lavrent'ev [32],[34], Ivanov [17], Newman [41], Koroljuk [24], Maz'ya and Havin [37], Jarmuhamedov [18], Shlapunov [55], and others. For holomorphic functions of one variable the Cauchy problem was considered in the papers of Carleman [8], Zin [66], Fok and Kuny [12], Patil [42], Krein and Nudelman [26], Steiner [59], and by other mathematicians. The Cauchy problem for the overdetermined Cauchy-Riemann system was studied by Tarkhanov [62], Znamenskaya [67], Aizenberg and Kytmanov [3], Karepov and Tarkhanov [19], Karepov [21], Shlapunov and Tarkhanov [52],[53],[54], and others. The question of the regularization of the Cauchy problem for the system of elasticity theory in space was studied by Mahmudov [36]. The Cauchy problem for general systems of differential equations with injective symbols has been investigated by Tarkhanov [61]-[64], Nacinovich [40], and others.

What place does our paper occupy among those cited ? If we try to answer this question we can say it is an attempt to elucidate new facts that the application of bases with double orthogonality brings to the Cauchy problem for general systems of differential equations with injective symbols (see Slepian and Pollak [56], Landau and Pollak [29]-[30], Slepian [57]).

As to the results, we should like to comment upon two facts. Firstly, the solvability conditions obtained are constructive, and simpler and more convenient than those known so far (see Tarkhanov [62]). Secondly, a constructive formula for the regularization (approximate solution) of the Cauchy problem for general systems of differential equations with injective symbols has been devised. Earlier it was known that such a regularization (Carleman's formula) existed (see Tarkhanov [61])). But the hope for simplicity and a constructive approach existed only for the Cauchy-Riemann system, or, more generally, systems factorizing the Laplace operator (see Aizenberg [1], Jarmuhamedov [18], Mahmudov [36], and others).

In §1 we elaborate the operator-theoretical foundations of the application of bases with double orthogonality to the problem of the continuation of classes of functions from massive subsets to the whole set. In a paper dated 1927 Bergman (see [6], p.14-20) developed the remarkable concept of the consequence of analytic functions. These functions are orthogonal in pairs with respect to integration over two domains one of which contains the closure of the other. He used this idea, at least in principle, to study the criterion of analytic extension. This beautiful and potentially useful idea did not receive sufficient recognition, probably because its practical application requires the preliminary solution of an eigenvalue problem, which may be difficult to solve. The idea of bases with double orthogonality appeared again in a series of the papers by Slepian and Pollak [56], Landau and Pollak [29]-[30], and Slepian [57]) in the sixties independently of Bergman. Shapiro [49] is sure that Bergman knew well that the phenomenon of double orthogonality had a more general character than being simply a fragment of the study of analytic functions. These abstract components are none other than the spectral theorem for a compact self-adjoint operator which is sometimes credited to F. Riesz (see Riesz and Sz.- Nagy [46], s. 93). Krasichkov [25] has shown that the use of the spectral theorem leads quite simply to an abstract Bergman theorem about the existence of bases with double orthogonality (see also Shapiro [49],[50]). Our account in §1 reproduces Bergman's concept in general, except that we considering continuous systems of functions with double orthogonality.

As Problem 1 may be unsolvable even in the class of all smooth (vector-) functions f in D (not only those satisfying $Pf = 0$) there are formal difficulties in the setting of the problem. To remove these difficulties it is necessary that the sections $f_\alpha (|\alpha| \leq p-1)$ should be restrictions to S of the corresponding derivatives of some smooth section in D. This is connected with the correct setting of the Cauchy problem which corresponds to a suitable Green's formula for solutions. The relevant results are described in §2.

In §3 a solvability criterion for the Cauchy problem for elliptic systems in the Hardy class $H^2(D)$ (see Tarkhanov [62]) is deduced in terms of bases with double orthogonality on the boundary of D. The corresponding eigenvalue problem is associated with a non-compact operator. Surface bases with double orthogonality are continuous systems of generalized eigenvectors of this operator (see Berezanskii [5], ch. V). Surface bases with double orthogonality in the Cauchy problem for holomorphic functions of one variable seemed to have been first applied by Krein and Nudelman [26]. Theorems on the jump of an integral of Green's type with density of this or that class imply that the behaviour of a solution f of Problem 1 near S is completely determined by the smoothness of the Cauchy data f_α $(|\alpha| \leq$

$p-1$). In particular, if $f_\alpha \in C^{p-1-|\alpha|}(E_{|\overset{\circ}{S}})$ (where $\overset{\circ}{S}$ is the interior of S in ∂D) then $f \in C^{p-1}_{loc}(S \cup D)$ (see Tarkhanov [63]). As for the behaviour of f near some other part of the boundary of D, it is determined by that class of functions (sections) in which we seek the solution of the Cauchy problem. The application of space bases with double orthogonality dictates the class that a solution belongs to. In fact it is one of the Sobolev spaces $W^{s,2}(E_{|D})$. In §4 we investigate weak limit values on the boundary of the domain D for solutions of systems with injective symbols in the Sobolev class $W^{s,q}(E_{|D})$. As a matter of fact, we present another view on the results of Rojtberg [47] about values on the boundary of generalized solutions of elliptic equations.

In §5 we prove a solvability criterion for the Cauchy problem for elliptic systems in terms of the Green integral. Using the Cauchy data on S we construct a Green integral satisfying $Pf = 0$ everywhere outside of S. Then the Cauchy problem is solvable if and only if this integral continues across S from the complement of D to this domain as a solution of the system $Pf = 0$ ($\in W^{s,q}(E_{|D})$). Although it is possible to obtain interesting examples directly from this, this result has an auxiliary character. In spite of the simplicity of the idea, its proof is complicated by some necessary facts from pseudo-differential operator theory on a manifold with boundary. For example, one of these facts is the theorem on the boundedness of potential operators in Sobolev spaces which was proved not long ago (see Eskin [11], Rempel and Schulze [45] and others).

In §6 the extendibility condition (as a solution of the system $Pf = 0$) across S of the Green integral is expressed in terms of space bases with double orthogonality. Its construction is connected with the solution of an eigenvalue problem for a certain compact operator, so this part of the application of bases with double orthogonality is most similar to the concept of Bergman [6]. We note that these ideas were tested on the example of the Cauchy problem for holomorphic functions (see the authors' article [51]) and we find some hints in the considerations of Aizenberg and Kytmanov [3].

The use of bases with double orthogonality not only gives information about solvability conditions for the Cauchy problem, but leads to explicit formulae for the regularization. A Carleman function of the Cauchy problem for solutions of elliptic systems is constructed in §7.

Finally, in §8 we consider some examples of differential equations of the simplest type. These are systems of the first order differential equations which are matrix factorizations of the Laplace equation. A system of homogeneous polynomials in $\mathbb{R}^n$ possessing the double orthogonality property relative to integration over every ball with centre at zero is constructed. Using it we obtain the solvability condition in an explicit form and obtain a formula for the regularization of the Cauchy problem for the simplest type systems in the special case. More exactly, S is a smooth hypersurface in a ball B with centre at zero, and D is that one of the two domains obtained by dividing B by S which does not contain the centre of the ball. The theorems on the solvability of the Cauchy problem and on the Carleman formula for holomorphic functions of one variable obtained in this way are the simplest ones (see Aizenberg and Kytmanov [3]). æ

PART I.

ELLIPTIC SYSTEMS

§1. Bases with double orthogonality.

As Shapiro [49] has observed, Bergman's problem is a special case of the question of when a given element of a Hilbert space belongs to the image of some injective compact operator with dense image.

In practice this problem appears usually in the following way. There is some linear continuous mapping of Hilbert spaces, $T : H_1 \to H_2$, say. Further, in H_1 a closed subspace Σ_1 is distinguished by some considerations. It is very helpful when the image of Σ_1 by the mapping T is closed in H_2. However this is not usually the case. In any case we denote by Σ_2 the closure of this image. Hence Σ_2 also is a Hilbert space with the hermitian structure induced from H_2.

Problem 1.1. *Let $h_2 \in \Sigma_2$. It is required to find a vector $h_1 \in \Sigma_1$ such that $Th_1 = h_2$.*

Except in trivial cases Problem 1.1 is incorrect. Therefore we can repeat the words which have been written in connection with these problems in the paper by one of the authors [62]. At the same time, the use of bases with double orthogonality gives a more satisfactory approach to Problem 1.1. We describe this.

We denote by Π the operator of the orthogonal projection on Σ_1 in H_1, and by M the operator T^*T in H_1, where $T^* : H_2 \to H_1$ is the mapping adjoint to the mapping T according to the theory of Hilbert spaces.

Proposition 1.1. *The restriction of the mapping ΠM to Σ_1 is a bounded linear operator from Σ_1 to Σ_1.*

Proof. In fact, the norm of the operator ΠM is not greater than $m = \|T\|^2$ even in H_2. $\square$

Proposition 1.2. *The operator $\Pi M : \Sigma_1 \to \Sigma_1$ is self-adjoint.*

Proof. The restriction to Σ_1 of the operator ΠM coincides with the restriction to this space of the (evidently) self-adjoint operator $\Pi M \Pi$. $\square$

Proposition 1.3. *The spectrum of the operator* $\Pi M : \Sigma_1 \to \Sigma_1$ *belongs to the segment* $[0; m]$.

Proof. By Propositions 1.1 and 1.2 we can conclude that the spectrum of the operator ΠM belongs to the segment $[-m; m]$. On the other hand, this operator is non-negative, because for $h \in \Sigma_1$ we have

$$(\Pi M h, h)_{H_1} = (Mh, h)_{H_1} = \|Th\|^2_{H_2} \geq 0$$

This proves our statement. □

Problem 1.1 is definite if and only if the restriction of the operator T on Σ_1 is injective. A corresponding conclusion follows for the operator ΠM.

Proposition 1.4. *The mappings* $\Pi M : \Sigma_1 \to \Sigma_1$ *and* $T : \Sigma_1 \to \Sigma_2$ *are simultaneously injective or not injective.*

Proof. It is sufficient to prove that the kernels of these operators coincide. However, for $h \in \Sigma_1$, $\Pi M h = 0$ if and only if $(Mh, g)_{H_1} = (Th, Tg)_{H_2} = 0$ for all $g \subset \Sigma_2$, that is, if and only if $Th = 0$. This proves the proposition. □

We can apply now the spectral theory of self-adjoint operators (see Riesz and Sz.-Nagy [46], s. 107). Namely, let E_λ $(-\infty < \lambda < \infty)$ be an orthogonal decomposition of the unit in the Hilbert space Σ_1 corresponding to the operator ΠM. In the simplest case of a discrete spectrum $\lambda_1, \lambda_2, ...$ we have $E_\lambda = \sum_{\lambda \leq \lambda_j} pr_{\lambda_j}$ where pr_{λ_j} is the orthogonal projection to the eigen subspace of ΠM corresponding to the eigenvalue λ_j. In the general case E_λ is some family of orthogonal projections concentrated on the spectrum of ΠM, and growing from 0 to I while λ changes from $-\infty$ to $+\infty$. This family has certain well known properties.

Theorem 1.5 (abstract Bergman's theorem). *Problem 1.1 is solvable if and only if*

$$\int_{-0}^{m} \frac{1}{\lambda^2} d(E_\lambda \Pi T^* h_2, \Pi T^* h_2)_{H_1} < \infty. \tag{1.1}$$

Proof. The condition (1.1) means that the vector $\Pi T^* h_2 \in \Sigma_1$ belongs to the domain of the (left) inverse operator of the operator $\Pi M : \Sigma_1 \to \Sigma_1$. Hence one can find an element $h_1 \in \Sigma_1$ such that $\Pi M h_1 = \Pi T^* h_2$. This implies that the vector $Mh_1 - T^* h_2 = T^*(Th_1 - h_2)$ is orthogonal to the subspace Σ_1 in H_1. In other words we have $(T^*(Th_1 - h_2), g)_{H_1} = (Th_1 - h_2), Tg)_{H_2} = 0$ for all $g \in \Sigma_1$. Under the hypothesis, the vector h_2 belongs to the closure of the image of the mapping $T : \Sigma_1 \to \Sigma_2$. This means that one can find a sequence $\{f_j\} \subset \Sigma_2$ such that Tf_j converges to h_2 in H_2. Hence

$$\|Th_1 - h_2\|^2_{H_2} = \lim_{j \to \infty} (Th_1 - h_2, T(h_1 - f_j))_{H_2} = lim_{j \to \infty} 0 = 0,$$

therefore $Th_1 = h_2$. Thus, we see that the equalities $\Pi M h_1 = \Pi T^* h_2$ and $Th_1 = h_2$ are equivalent. This completes the proof of the theorem. $\square$

From the proof of Theorem 1.5 one can see a curious phenomenon. Namely, if Problem 1.1 is solvable then its solution is unique. The formula for this solution is given in the following theorem.

THEOREM 1.6 (ABSTRACT CARLEMAN'S FORMULA). *Under condition (1.1) a solution of Problem 1.1 is given by the formula*

$$h_1 = \int_{-0}^{m} \frac{1}{\lambda} d(E_\lambda \Pi T^* h_2). \tag{1.2}$$

PROOF. Condition (1.1) guarantees the convergence of integral (1.2) in the weak topology of the space Σ_1. Therefore $h_1 \in \Sigma_1$ and we need only prove that $\Pi M h_1 = \Pi T^* h_2$. Now

$$\Pi M h_1 = \int_0^m \lambda \frac{1}{\lambda} d(E_\lambda \Pi T^* h_2) = \int_{-0}^m d(E_\lambda \Pi T^* h_2) = \Pi T^* h_2,$$

which was to be proved. $\square$

We emphasize once again that under condition (1.1) the integral in formula (1.2) converges in the weak topology of the space Σ_1.

If we use the representation of the projections E_λ $(-\infty < \lambda < \infty)$ by means of the eigen vectors of the operator $\Pi M : \Sigma_1 \to \Sigma_1$ (see Berezanskii [5]. ch. V) then we can see that it is possible to make formulae (1.1) and (1.2) more visible. For let $L_1 \subset \Sigma_1 \subset L_1'$ where L_1 is a topological vector space such that the embedding $L_1 \subset \Sigma_1$ is quasi-kernel, and the operator ΠM admits an extension $\Pi M : L_1 \to L_1$. Having taken the transposed mapping to this mapping we obtain a continuation of ΠM to a continuous linear operator on L_1' which is denoted by $\widetilde{\Pi M}$. Under the above assumption on L_1, the operator $\widetilde{\Pi M}$ has a complete system of generalized eigenvectors $\{b_\lambda^{(i)}\}_{\lambda \in \mathbb{R}}^{1 \le i \le n_\lambda}$ in L_1' (see Berezanskii [5], p.341). This means that $\widetilde{\Pi M} b_\lambda^{(i)} = \lambda b_\lambda^{(i)}$, and for any vectors $h, g \in L_1$ there is Parseval's equality

$$(E(\Delta)h, g)_{H_1} = \int_\Delta \sum_{i=1}^{n_\lambda} (h, b_\lambda^{(i)})_{H_1} \overline{(g, b_\lambda^{(i)})_{H_1}} d\sigma(\lambda).$$

Here $E(\Delta) = \int_\Delta dE_\lambda$ is the spectral measure corresponding to the operator ΠM, and $d\sigma(\lambda)$ is a nonnegative Borel measure on the real axis. Using Parseval's equality for vectors from L_1 one can extend the "Fourier transformation" $(h, b_\lambda^{(i)})_{H_1}$ to vectors from Σ_1 by continuity. Then we have (in the sense of the $*$-weak convergence of the integrals in L_1')

$$E_\lambda h = \int_{-\infty}^{\lambda} \sum_{i=1}^{n_\lambda} (h, b_\zeta^{(i)})_{H_1} b_\zeta^{(i)} d\sigma(\zeta) \ \ (h \in \Sigma_1). \tag{1.3}$$

COROLLARY 1.7 (ABSTRACT BERGMAN'S THEOREM). *Problem 1.1 is solvable if and only if*

$$\int_{-0}^{m}\sum_{i=1}^{n_\lambda}\left|\frac{(\Pi T^* h_2, b_\lambda^{(i)})_{H_1}}{\lambda}\right|^2 d\sigma(\lambda)<\infty. \tag{1.4}$$

PROOF. Using the equality (1.3), we obtain

$$d(E_\lambda \Pi T^* h_2, \Pi T^* h_2) = d\int_{-\infty}^{\lambda}\sum_{i=1}^{n_\zeta}|(\Pi T^* h_2, b_\zeta^{(i)})_{H_1}|^2 d\sigma(\zeta) =$$

$$\sum_{i=1}^{n_\lambda}|(\Pi T^* h_2, b_\lambda^{(i)})_{H_1}|^2 d\sigma(\lambda).$$

In view of Theorem 1.5, we obtain the statement of the corollary. □

COROLLARY 1.8 (ABSTRACT CARLEMAN'S FORMULA). *Under condition (1.1) a solution of Problem 1.1 is given by the following formula (where convergence is understood in the $*$-weak topology of the space L_1') :*

$$h_1 = \int_{-0}^{m}\sum_{i=1}^{n_\lambda} b_\lambda^{(i)}\frac{(\Pi T^* h_2, b_\lambda^{(i)})_{H_1}}{\lambda} d\sigma(\lambda). \tag{1.5}$$

PROOF. It is sufficient to calculate

$$dE_\lambda(\Pi T^* h_2) = \sum_{i=1}^{n_\lambda} b_\lambda^{(i)}(\Pi T^* h_2, b_\lambda^{(i)})_{H_1} d\sigma(\lambda).$$

and to put it in formula (1.2). □

We consider an instructive example.

EXAMPLE 1.9. We suppose that the operator $T : \Sigma_1 \to \Sigma_1$ is 1) injective, 2) compact. Then, by Proposition 1.4 the operator $\Pi M : \Sigma_1 \to \Sigma_1$ is injective, and (the compactness of T and) the boundedness of ΠT^* implies that $\Pi M : \Sigma_1 \to \Sigma_1$ is compact. According to the spectral theorem for compact self-adjoint operators (see Riesz and Sz.-Nagy [46], s. 93), ΠM has in Σ_1 a countable complete system of eigenvectors $\{b_j\}_{j=1}^{\infty}$ corresponding to positive eigenvalues $\{\lambda_j\}$. However simple calculations show that $(Tb_j, Tb_j)_{H_2} = \lambda_j (b_j, b_j)_{H_1}$, that is, the system $\{Tb_j\}$ is orthogonal in Σ_2. Evidently this system is complete in Σ_1, hence it gives an orthogonal basis in this space. We notice that the system $\{b_j\} \subset \Sigma_1$ possesses the double orthogonality property : 1) relative to the scalar product $(.,.)_{H_1}$ in Σ_1

and 2) relative to the scalar product $(T.,T.)_{H_2}$ in Σ_2. As we noted in the introduction, Bergman was the first to devise these systems (see [6]), and Krasichkov [25] proved the abstract existence theorem. The orthogonal decomposition of the unit corresponding to the operator $\Pi M : \Sigma_1 \to \Sigma_1$ is now given by the operators $E_\lambda h = \sum_{\lambda \le \lambda_j} b_j (h, b_j)_{H_1}$ (see (1.3)). Relations (1.4) and (1.5) take the form $\sum_{j=1}^{\infty} |c_j|^2 < \infty$ and $h_1 = \sum_{j=1}^{\infty} c_j b_j$ respectively, where $c_j = \frac{(h_2, Tb_j)_{H_2}}{\|Tb_j\|^2_{H_2}}$ are Fourier coefficients of the vector $h \in \Sigma_2$ relative to the orthogonal system (basis) $\{Tb_j\}$ in this space. □

In the general case a system $\{b_\lambda^{(i)}\}$ also keeps some properties of bases with double orthogonality. We describe now an alternative method for its construction, using this idea. In the following we shall not take enough care of the legality of operations, because we want to make clear the idea only. The problem is first to construct a basis in Σ_2 and then to obtain by means of it a basis in Σ_1. We consider the operator $T\Pi T^* : \Sigma_2 \to \Sigma_2$. Again we notice that it is a bounded self-adjoint operator with the same spectrum, as ΠM. This operator is always injective, and it inherits the compactness property from $T : \Sigma_1 \to \Sigma_2$. We notice that the mapping $\Pi T^* : \Sigma_2 \to \Sigma_1$ is adjoint to $T : \Sigma_1 \to \Sigma_2$ in the sense of Hilbert spaces. To describe the image of T one can use an orthogonal decomposition of the unit $\{I_\lambda\}$ in Σ_2 corresponding to the operator $T\Pi T^*$. Then the solvability condition for Problem 1.1 has the form $\int_{-0}^{m} \frac{1}{\lambda} d(I_\lambda h_2, h_2) < \infty$, and the solution is given by the formula $h_1 = \Pi T^* \int_{-0}^{m} \frac{1}{\lambda} dI_\lambda(h_2)$. F urther, the projection operators I_λ can be presented, similarly to (1.3), by generalized eigen vectors of the operator $T\Pi T^*$ in L_2', where $L_2 \subset \Sigma_2 \subset L_2'$ is a suitable equipment of the Hilbert space Σ_2. Let $\{e_\lambda^{(i)}\}$ be a complete system of these vectors in L_2'. Then, if the operator T is injective, $\{b_\lambda^{(i)}\}$ (where $b_\lambda^{(i)} = \frac{1}{\lambda} \Pi T^* e_\lambda^{(i)}$) is a complete system of generalized eigen vectors of the operator ΠM. We leave the reader to write the formulae, similar to (1.4) and (1.5), in terms of the system $\{e_\lambda^{(i)}\}$.

Example 1.10. Krein and Nudelman [26] have considered the Cauchy problem for holomorphic functions of the Hardy class H^2 in the lower half-plane with Cauchy data on the segment $[-1;1]$ of the real axis. They had $H_1 = L^2(\mathbb{R}^1)$, $H_2 = L^2([-1;1])$, the Hardy space Σ_1, and the operator of restriction $T : \Sigma \to H_2$. In this case we have $\Sigma_2 = H_2$. The projection $\Pi : H_1 \to \Sigma_1$ is given by means of limit values on $\mathbb{R}^1$ of the Cauchy type integral in the lower half-plane. The operator $T\Pi T^* : \Sigma_2 \to \Sigma_2$ is an integral operator (but it is not the Carleman operator) with a simple spectrum. The complete system of generalized eigenfunctions of this operator was earlier constructed by Koppelman and Pincus [23]. Having extrapolated it by the operator ΠT^* on the whole real axis, Krein and Nudelman [26] obtained a continuous system of functions with double orthogonality in Σ_1. They also indicated a solvability condition, and a formula for the regularization of the Cauchy problem. □

We finish this section with one more example connected with the Cauchy problem for holomorphic functions when the support of the Cauchy data is a "thin" set.

EXAMPLE 1.11. Let σ be a compact set of positive measure in $\mathbb{R}^n$. We denote by W_σ the set of Fourier transforms of functions from $L^2(\sigma)$, that is, the set of functions of the type $\hat{f}(\zeta) = \frac{1}{(2\pi)^n}\int_\sigma e^{i\zeta x} f(x)dx$, where $f \in L^2(\sigma)$. According to the theorem of Paley and Wiener, elements of W_σ are restrictions on $\mathbb{R}^n$ of (not all!) entire functions of exponential order of growth in $\mathbb{C}^n$. For this reason W_σ is called the Wiener class. By means of the Plancheral theorem it is easy to see that W_σ is a closed subset of $L^2(\mathbb{R}^n)$. Let $S \subset \mathbb{R}^n$ be a given bounded set with a non-negative Borel measure m. In order not to complicate the notation we use the symbol $L^2(S)$ for the space of (classes of) functions which are measurable and square-integrable relative to the measure m on S. As for the assumptions about (S, m), we require that restrictions to S of (infinitely) differentiable functions in $\mathbb{R}^n$ should be contained in $L^2(S)$, and dense in this space. We consider the following problem: for a given function $f_0 \in L^2(S)$, find a function $f \in W_\sigma$ such that $f_{|S} = f_0$. To include it in the general scheme of Problem 1.1 we set $H_1 = \Sigma_1 = W_\sigma$, $H_2 = L^2(S)$, and define the operator $T : H_1 \to H_2$ as the restriction of functions on S. One can show that the operator T has a dense image. For let Φ be a continuous linear functional on $L^2(S)$ which vanishes on the image of T. According to the Riesz theorem, there is a function $\varphi \in L^2(S)$ such that $\Phi(f) = \int_S f\varphi dm$ for all $f \in L^2(S)$. Then one can consider Φ in explicit form as a distribution with compact support in $\mathbb{R}^n$. The condition $\Phi_{|imT} = 0$ implies that the Fourier transform $\hat{\Phi}$ of the distribution Φ vanishes on σ. Since $\hat{\Phi}$ is an entire function, and the measure of σ is positive then $\hat{\Phi} \equiv 0$ everywhere in $\mathbb{R}^n$. From this we conclude that Φ is the zero distribution in $\mathbb{R}^n$, that is, the zero functional on $L^2(S)$. Hence in our case we have $\Sigma_2 = H_2$. It is not difficult to verify that the operator T is compact. We shall assume its injectivity, in order that the Cauchy problem be defined. This simply means that S is a set of uniqueness for the class W_σ. Then we have the situation considered in Example 1.9. According to our earlier conclusions, if we denote by $\{b_j\}$, $j = 1, 2, ...$, a complete orthonormal system of eigenvectors of the operator T^*T in W_σ then the systems $\{Tb_j\}$, $j = 1, 2, ...$, will be an orthogonal basis in $L^2(S)$. The condition of solvability and the formula for the regularization of solutions of the Cauchy problem have the forms $\sum_{j=1}^\infty |c_j|^2 < \infty$ and $f = \sum_{j=1}^\infty c_j b_j$ respectively, where $c_j = \frac{(f_0, Tb_j)_{L^2(S)}}{\|b_j\|^2_{L^2(S)}}$ are Fourier coefficients of the function f relative to the orthogonal system $\{Tb_j\}$ in $L^2(S)$. If S is a set of positive measure in $\mathbb{R}^n$, then the results of this example were obtained by Krasichkov [25]. □

æ

§2. The Cauchy problem for solutions of systems with injective symbols

We suppose that $D \Subset X$ is a domain with a boundary of class C^p (for $p = 1$ it is required that $\partial D \in C^2$). For some of the results of the paper a higher smoothness of the boundary is required, but always it is sufficient that $\partial D \in C^\infty$.

We define the function $\rho(x)$ by $\pm dist(x, \partial D)$ where the sign "$-$" corresponds to the case $x \in D$, and "$+$" to the case $x \in X \setminus \overline{D}$. Then, if a neighbourhood U of the boundary ∂D is sufficiently small, $\rho \in C^p_{loc}(U)$, and $|d\rho| = 1$ in U.

Hence, for small $|\varepsilon|$, the domains $D_\varepsilon = \{x \in D : \rho(x) < -\varepsilon\}$ have boundaries of the class C^p, and as $\varepsilon \to +0(-0)$ they approximate D from inside (outside). Here the unit outward normal vector $\nu(x)$ to the surface ∂D at the point x is given by the gradient $\nabla\rho(x)$. The inner product $ds = \nabla\rho \rfloor dv$ provides the volume form induced by the volume $dv (= dx)$ on X on every surface ∂D_ε.

We fix a Dirichlet system of order $(p-1)$ on ∂D, say, $B_j \in do_{b_j}(E \to G_j)$ $(0 \leq j \leq p-1)$ where $G_j = U \times C^k$ are (trivial) bundles in U. The words "Dirichlet system of order $(p-1)$ on ∂D" mean that 1) system B_j is normal, that is, the orders of the differential operators are pairwise different, and each of the mappings $\sigma(B_j)(x, \nabla\rho(x))$ is surjective for all $x \in \partial D$, 2) $b_j \leq p-1$ for all j (see Berezanskii [5], p.233).

We use the system of boundary operators $\{B_j\}$ to reformulate Problem 1 in the following form.

PROBLEM 2.1. *Let f_j $(0 \leq j \leq p-1)$ be sections of the bundles G_j over the set S. It is required to find a solution $f \in S(D)$ such that the expressions $B_j f$ $(0 \leq j \leq p-1)$ have in a suitable sense limit values on S coinciding with f.*

In order to justify the term "the Cauchy problem" for Problem 2.1, we note that the values of $B_j f$ $(0 \leq j \leq p-1)$ on S determine all the derivatives of f up to order $p-1$ on S. At the same time Problem 2.1 is solvable in the class of smooth (vector-) functions f, that is, it is not necessary to think about formal agreements between the sections f_j $(0 \leq j \leq p-1)$.

The weak limit values $B_j f$ $(0 \leq j \leq p-1)$ on ∂D are most important for applications. We distinguish the maximal class of solutions f for which one can speak of these limit values.

DEFINITION 2.2. *The space $S_{P,B}(D)$ consists of all solutions $f \in S(D)$ for which the expressions $B_j f$ $(0 \leq j \leq p-1)$ have weak limit values $f_j \in D'(G_{j|\partial D})$ on ∂D in the following sense*

$$\lim_{\varepsilon \to 0} \int_{\partial D} < g, B_j f(x - \varepsilon\nu(x)) > ds = \int_{\partial D} < g, f_j > ds \ for \ all \ g \in C^\infty_{comp}(G^*_{j|\partial D}).$$

It is clear that, if $f \in S(D) \cap C^{p-1}(E_{|\overline{D}})$, the weak boundary values of the expressions $B_j f$ $(0 \leq j \leq p-1)$ on ∂D exist and coincide with the usual restrictions $B_j f$. In order to relate the weak limit values of $B_j f$ $(0 \leq j \leq p-1)$ on ∂D to other (radial, non-tangential, in some norm) limits, the Green formula and theorems on the jump of the boundary integral in this formula are usually used. The construction of the Green formula is based on the following lemma.

LEMMA 2.3. *If the neighbourhood U is sufficiently small, there is a Green operator G_P for the differential operator P in U which has the following form*

$$G_P(g,f) = \sum_{j=0}^{p-1} < C_j g, B_j f >_x \, ds \, + \, \frac{d\rho}{|d\rho|} \Lambda G_\nu(g,f)$$

where $C_j \in do_{p-1-b_j}(F^* \to G_j^*)$ $(0 \le j \le p-1)$ is some Dirichlet system of order $(p-1)$ on ∂D, and $G_\nu \in do_{p-1}((F^*,E)_{|U} \to \Lambda^{n-2})$.

PROOF. See Tarkhanov [63], Lemma 28.3. □

We have taken a liberty in wording Lemma 2.3. Namely, according to the usual understanding, differential operators on X must have (infinitely) differentiable coefficients, however the smoothness of the coefficients of the differential operators $\{C_j\}$ and G_ν is finite. One may check what smoothness requirements for the coefficients of $\{C_j\}$ are satisfied as a consequence of the supposed smoothness of the boundary of D (and coefficients of the initial expressions $\{B_j\}$). Certainly, these difficulties are removed if $\partial D \in C^\infty$. For our purposes it is sufficient that the coefficients of every differential operator B_j belong to the class $C^{p-1-b_j}_{loc}$, and the coefficients of each differential operator C_j belong to the class C^{b_j} in the neighbourhood U.

Since the differential operator $P(= P^0)$ satisfies the condition $(U)_S$ (see the introduction), the complex $\{E^i, P^i\}$ has a fundamental solution in degree 0, say, $\{\Phi^i\}$, $\Phi \in pdo_{-p_{i-1}}(E^i \to E^{i-1})$ where $pdo_m(E^i \to E^{i-1})$ is the vector space of the all pseudo- differential operators of type $(E^i \to E^{i-1})$ and order m (see Tarkhanov [63], Corollary 27.8). This means that $\Phi^{i+1}P^i + P^{i-1}\Phi^i = 1 - S^i$ on $C^\infty_{comp}(E^i)$ where $S^i \in pdo_{-\infty}(E^i \to E^i)$ are smoothing operators, and $S^0 = 0$. In particular, the component $\Phi = \Phi^1$ is a left fundamental solution of the differential operator P.

THEOREM 2.4. *For any solution $f \in S_{P,B}(D)$ we have the Green formula*

$$- \int_{\partial D} \sum_{j=0}^{p-1} < C_j \Phi(x,y), B_j f >_y \, ds = \begin{cases} f(x), x \in D, \\ 0, x \in X \backslash \overline{D}. \end{cases} \tag{2.1}$$

PROOF. First, the theorem of Banach and Steinhaus implies that, for a solution $f \in S(D)$, the expressions $B_j f$ $(0 \le j \le p-1)$ have weak limit values $f_j \in D'(G_{j|\partial D})$ on ∂D if and only if

$$\lim_{\varepsilon \to +0} \int_{\partial D_\varepsilon} < g, B_j f >_x \, ds = \int_{\partial D} < g, f_j >_x \, ds \; for \, all \, g \in C^\infty_{comp}(G_j^*). \tag{2.2}$$

We now choose a number $\varepsilon > 0$ so small that $\partial D_\varepsilon \subset U$. We represent the solution $f \in S(D)$ in the domain D by the Green formula, having taken as Green's operator of the differential operator P the operator from Lemma 2.3. Then, since the restriction of the differential $d\rho$ on the surface ∂D is equal to zero, we get formula (2.1) where in place of D we have the domain D. Having made the limit passage by $\varepsilon \to +0$, and having used equality (2.2) we obtain the theorem. □

Formula (2.1) gives the apparatus for the effective control of the heuristic consideration that the behaviour of a solution $f \in S_{P,B}(D)$ near a point $x \in \partial D$ in the closure of the domain is completely determined by the "smoothness" property near x on ∂D of the weak boundary values $B_j f$ $(0 \le j \le p-1)$. Thus for $f \in D'(G_{j|\partial D})$ $(0 \le j \le p-1)$ we set $f = \oplus f_j$ so that $f \in D'(\oplus G_{j|\partial D})$, and

$$\mathcal{G} f(x) = \int_{\partial D} \sum_{j=0}^{p-1} < C_j(y)\Phi(x,y), f >_y ds \ (x \notin \partial D).$$

Let $\mathcal{N}$ be a relatively compact neighbourhood of the point x in X, and $\varphi_\varepsilon \in C^\infty_{comp}(X)$ be a function supported on the ε-neighbourhood of $\mathcal{N}$ and beying equal to 1 in $\mathcal{N}$. Then, denoting by χ_D the characteristic function of the domain D, we can rewrite formula (2.1) in the form $\chi_D f = -\mathcal{G}(\varphi_\varepsilon(\oplus B_j f)) - \mathcal{G}(1-\varphi_\varepsilon)(\oplus B_j f))$. The first summand in (2.1) depends only on the values of $B_j f$ $(0 \le j \le p-1)$ in the ε-neighbourhood of the set $\mathcal{N} \cap \partial D$ on the boundary, and the second one is an infinitely differentiable section of E in N. Hence, the character of "the transition" of the solution f from $\mathcal{N} \cap D$ to its weak limit values on $\mathcal{N} \cap \partial D$ is completely determined by the jump behaviour of the surface integral $\mathcal{G}(\varphi_\varepsilon(\oplus B_j f))$ in going across $\mathcal{N} \cap \partial D$. This integral is called the Green integral of the (vector-value) distribution $\varphi_\varepsilon(\oplus B_j f)$.

Corollary 2.5. *If for a solution $f \in S_{P,B}(D)$ we have $B_j f \in C^{p-B_j-1}_{loc}(G_{j|\overset{\circ}{S}})$ $(0 \le j \le p-1)$ then $f \in C^{p-1}_{loc}(E_{|D\cup \overset{\circ}{S}})$.*

Proof. Since differentiability is a local property then, as we said above, it is sufficient to consider the case $S = \partial D$. According to Lemma 28.2 of Tarkhanov [63], we can find a section $\hat{f} \in C^{p-1}_{loc}(E)$ such that $B_j \hat{f} = B_j f$ $(0 \le j \le p-1)$ on ∂D. Then Theorem 2.4 and Lemma 2.3 imply that $\chi_D f = -\int_{\partial D} G_P(\Phi(x,y), \hat{f}(y))$. In particular, the integral $\int_{\partial D} G_P(\Phi(x,y), \hat{f}(y))$, being considered for $x \in X \backslash \overline{D}$, is equal to zero. Therefore it extends continuously together with its derivatives up to order $(p-1)$ to the closure of $X \backslash \overline{D}$. But then, from Lemma 29.5 (Tarkhanov [63]), it is easy to show that (see, for example, Lemma 1.1 in the paper of Shlapunov [55]) the integral $\int_{\partial D} G_P(\Phi(x,y), \hat{f}(y))$ $(x \in D)$ extends continuously together with its derivatives up to order $(p-1)$ to the closure of D. Hence $f \in C^{p-1}_{loc}(E_{|\overline{D}})$, which which was to be proved. □

In Definition 2.2 of the space $S_{P,B}(D)$ we used a Dirichlet system $\{B_j\}$, and it seems that the set of elements of $S_{P,B}(D)$ depends essentially on the choice of this system. The fact that this is not so is unexpected. We shall say that a solution $f \in S(D)$ has finite order of growth near the boundary (∂D) if for any point $x^0 \in \partial D$ there are a ball $B(x^0, R)$, and constants $c > 0$ and $\gamma > 0$ such that $|f(x)| \leq c\, dist(x, \partial D)^\gamma$ for all $x \in B(x^0, R) \cap D$. In view of the compactness of ∂D, the constants c and γ can be chosen so that the estimate holds for all $x \in \partial D$. The following theorem for harmonic functions was proved by Straube [60].

THEOREM 2.6. *A solution $f \in S(D)$ belongs to $S_{P,B}(D)$ if and only if it has finite order of growth near ∂D.*

PROOF. Necessity. Any distribution on ∂D locally has finite order of singularity, and the kernel $\Phi(x, y)$ is infinitely differentiable everywhere outside of the diagonal $\{x = y\}$, and on the diagonal this kernel has the same type of singularity as the well known fundamental solution of $(p/2)$-th degree of the Laplace operator. So the necessity of the condition of the theorem follows from formula (2.1).

Sufficiency. Let $f \in S(D)$ have finite order of growth, say, γ, near the boundary. It is clear that together with $Pf = 0$ we have $P^*Pf = 0$ where P^* is (formally) adjoint to the differential operator P. The operator P^*P is an elliptic operator of order $2p$. We can complete the system $\{B_j\}_{j=0}^{p-1}$ to a Dirichlet system of order $(2p-1)$ on ∂D, say, $\{B_j\}_{j=0}^{2p-1}$, and then we can try to prove that any expression $B_j f$ $(0 \leq j \leq 2p-1)$ has a weak limit on ∂D according to Definition 2.2. When this is proved, we shall have obtained formally more than we require. Of course, it comes to the same thing, because the differential operator P and the system $\{B_j\}_{j=0}^{p-1}$ are arbitrary. So, without loss of generality, we can require that the differential operator P is elliptic. But we can not assume for P^*P the condition $(U)_S$ on X. Therefore for P one can only guarantee the existence of a parametrix $\Phi \in pdo_{-p}(F \to E)$, that is, in particular, $\Phi P = 1 - S^0$ for some smoothing operator $S \in pdo_{-\infty}(E \to E)$. We now consider this situation. Rojtberg [47] showed that one can naturally define a regularization $\hat{f}$ of the solution f as a continuous linear functional on the space $C^{s'}(E_{\overline{D}})$ for a suitable s' depending on the order of singularity of f near the boundary (γ). Then $\hat{f} = f$ in D, and $\hat{f} \in W^{-s,q'}(E_{|D})$ $(= W^{s,q}(E^*_{|D})'))$, where $s > \frac{n}{q} + (\gamma - 1)$, and $\frac{1}{q} + \frac{1}{q'} = 1$ $(q > 1)$. Further, for the solution f there are limit values of the expressions $B_j f$ $(0 \leq j \leq p-1)$ on ∂D, these being understood in the following sense. There is a sequence $f^{(\nu)} \in C^\infty(E_{|\overline{D}})$ such that $f^{(\nu)}$ converges to $\hat{f}$ in $W^{-s,q'}(E_{|D})$, and $Pf^{(\nu)}$ converges to zero in $W^{-s-p,q'}(F_{|D})$. Moreover, for any such sequence $f^{(\nu)}$ the sequences $B_j f^{(\nu)}$ $(0 \leq j \leq p-1)$ are fundamental in the spaces $B^{-s-p-b_j-\frac{1}{q'},q'}(G_{j|\partial D})$, and therefore they converge in these spaces to limits f_j. Rojtberg called these sections f_j $(0 \leq j \leq p-1)$ the limit values of the expressions $B_j \hat{f}$ (or equivalently of $B_j f$) on ∂D. Now we want to show that the sections f_j $(0 \leq j \leq p-1)$ are the weak limits of the expressions $B_j f$ in the sense of Definition 2.2. To this end we write for the sections $f^{(\nu)}$ the Green formula in the domain D, that is,

$$\chi_D f^{(\nu)} = -\mathcal{G}(\oplus B_j f^{(\nu)}) + \Phi(\chi_D P f^{(\nu)}) + S^0(\chi_D f^{(\nu)})$$

(see, for example, formula (9.13) in the book of Tarkhanov [65]). If we calculate the limits of the left and right hand side of this equality, for example in the weak topology of the space $D'(E_{|X\backslash\partial D})$ then we obtain

$$-\mathcal{G}(\oplus f_j) + S^0(\chi_D \hat{f}) = \begin{cases} f(x), x \in D, \\ 0, x \in X\backslash\overline{D}. \end{cases} \tag{2.3}$$

We have convinced ourselves that the solution f is represented by the limit values on the boundary of the expressions $B_j f$ $(0 \le j \le p-1)$ according to Rojtberg [47], and by the regularization $\hat{f}$ in D by Green formula (2.3). The second summand on the left hand side of this formula is an infinitely differentiable section of E everywhere on the set X. Therefore the result follows from the following lemma.

Lemma 2.7. *We suppose that $D \Subset X$ is a domain with an infinitely differentiable boundary, and $f_j \in D'(G_{j|\partial D})$ $(0 \le j \le p-1)$ are given sections on ∂D. Then, for all sections $g_j \in D(G^*_{j|\partial D})$ $(0 \le j \le p-1)$ we have*

$$\lim_{\varepsilon\to+0}\int_{\partial D} < g_j, B_j(\mathcal{G}(f))(x+\varepsilon\nu(x)) - B_j(\mathcal{G}(f))(x-\varepsilon\nu(x)) >_x ds = \int_{\partial D} < g_j, f >_x ds.$$

Proof. We fix a section $g_j \in D(G^*_{j|\partial D})$ and we find a section $g \in C^\infty_{loc}(F^*)$ such that $C_j g = g_j$, and $C_j g = 0$ for $i \ne j$ on ∂D. It is not difficult to construct such a section g, for example, using the formulae for the jumps in crossing ∂D of a Green type integral with a smooth density. Then using Lemma 2.3 we can write

$$\lim_{\varepsilon\to+0}\int_{\partial D} < g_j, [B_j(\mathcal{G}(f))(x+\varepsilon\nu(x)) - B_j(\mathcal{G}(f))(x-\varepsilon\nu(x))] >_x ds =$$

$$= \lim_{\varepsilon\to+0}[\int_{\partial D_{-\varepsilon}} \sum_{j=0}^{p-1} < C_j g, B_j(\mathcal{G} f) >_x ds - \int_{\partial D_\varepsilon} \sum_{j=0}^{p-1} < C_j g, B_j(\mathcal{G} f) >_x ds] =$$

$$= \lim_{\varepsilon\to+0}\int_{\partial(D_{-\varepsilon}\backslash D_\varepsilon)} G_P(g, \mathcal{G} f).$$

Repeating the considerations on p.291 in the book of Tarkhanov [63] we obtain that the last limit exists, and that it is equal to

$$\int_{\partial D} < C_j g, f_j >_x ds = \int_{\partial D} < g_j, f_j >_x ds,$$

which was to be proved. □

As one can see from the proof of the lemma, it holds also for a domain D with a boundary of finite, perhaps, very high degree of smoothness. The same considerations can be applied to the smoothness of the sections g_j in (2.4). These depend on the orders of singularity of the given sections f_j $(0 \le j \le p-1)$ which are finite since the surface ∂D is compact.

We can now complete the proof of Theorem 2.6. In fact, if $g \in D(G^*_{j|\partial D})$ where $0 \le j \le p-1$, then, from formula (2.3) and Lemma 2.7, we obtain

$$\lim_{\varepsilon\to+0}\int_{\partial D} < g, B_j f(x-\varepsilon\nu(x)) >_x ds =$$

$$= \lim_{\varepsilon\to+0}\int_{\partial D} < g, -B_j(\mathcal{G}(\oplus f_j))(x-\varepsilon\nu(x)) + B_j(S^0(\chi_D \hat{f})(x-\varepsilon\nu(x)) >_x ds =$$

$$= \lim_{\varepsilon\to+0}\int_{\partial D} < g, -B_j(\mathcal{G}(\oplus f_j))(x-\varepsilon\nu(x)) + B_j(S^0(\chi_D \hat{f})(x+\varepsilon\nu(x)) >_x ds =$$

$$= \lim_{\varepsilon\to+0}\int_{\partial D} < g, -B_j(\mathcal{G}(\oplus f_j))(x-\varepsilon\nu(x)) + B_j(\mathcal{G}(\oplus f_j))(x+\varepsilon\nu(x)) >_x ds =$$

$$= \int_{\partial D} < g_j, f_j >_x ds,$$

that is, $f \in S_{P,B}(D)$. Hence Theorem 2.6 is completely proved. □

We note that Lemma 2.7 is similar to the theorem on the weak jump of the Bochner - Martinelli integral which was proved by Chirka [9]. We denote by $S^f(D)$ the subspace of $S(D)$ which consists of solutions of finite order of growth near the boundary of D. As we have just proved, for any Dirichlet system of order $(p-1)$ on ∂D, say, $\{B_j\}$, we have $S^f(D) = S_{P,B}(D)$. For several reasons, it is convenient to consider the Cauchy Problem 2.1 in a subspace of $S^f(D)$. We indicate now a class of boundary sets S for which Problem 2.1 has no more than one solution in $S^f(D)$.

THEOREM 2.8. *Suppose that for a solution $f \in S^f(D)$ the boundary values $B_j f$ $(0 \le j \le p-1)$ vanish on a set $S \subset \partial D$ which has at least one interior point. Then $f \equiv 0$ in D.*

PROOF. Denote, as above, by $\mathcal{G}(\oplus B_j f)$ the integral on the left hand side of formula (2.1). Let $x^0 \in S$, and $B = B(x^0, r)$ be an open ball in X such that $B \cap \partial D \subset S$. We set $O = D \cup B$. Then $\mathcal{G}(\oplus B_j f) \in C^\infty_{loc}(E_{|O})$ satisfies $P\mathcal{G}(\oplus B_j f) = 0$ in the domain $O \subset X$, and it vanishes on the non-empty open subset $B \backslash D$ of this domain. Since the uniqueness property of the Cauchy problem in the small on X holds for P then $\mathcal{G}(\oplus B_j f) = 0$ in O. In particularly, $f \equiv 0$ in D, which was to be proved. □

æ

§3. A criterion of the solvability of the Cauchy problem for elliptic systems in terms of surface bases.

In [62] the maximal subclasses of $S^f(D)$ of solutions f, for which one can speak of the boundary values of the expressions $B_j f$ $(0 \le j \le p-1)$ on ∂D belonging to the range of usual (not generalized) sections of G_j, was distinguished. These are the so-called Hardy spaces $H^2_{P,B}(D)$ $(1 < q < \infty)$ which are modelled on the pattern of the classical Hardy spaces of holomorphic functions. One could say that $H^2_{P,B}(D)$ consists of all solutions $f \in S_{P,B}(D)$ for which the weak limit values of the expressions $B_j f$ $(0 \le j \le p-1)$ on ∂D belong to $L^2(G_{j|\partial D})$. In particular, with the topology induced by $L^2(\oplus G_{j|\partial D})$ the space $H^2_{P,B}(D)$ is a Hilbert space (see below). In this section we indicate an application of the abstract theory of §1 to the Cauchy Problem 2.1 in the Hardy class $H^2_{P,B}(D)$. So, let P be an elliptic differential operator whose transposed operator (P') satisfies the uniqueness condition for the Cauchy problem in the small on X. We consider the following problem.

Problem 3.1. *Let $f_j \in L^2(G_{j|S})$ $(0 \le j \le p-1)$ be known sections on S. It is required to find a solution $f \in H^2_{P,B}(D)$, satisfying $B_j f = f_j$ $(0 \le j \le p-1)$ on S.*

As was noticed by M.M. Lavrent'ev, the fundamental result about the solvability of Problem 3.1 is the following.

Lemma 3.2. *If the complement of S on ∂D has at least one interior point then Problem 3.1 is densely solvable.*

Proof. We denote by H the vector space $L^2(\oplus G_{j|S})$. Having provided each of the bundles G_j with some Hermitian metric $(.,.)_x$ we can define the conjugate linear isomorphism $* : G_j \to G_j^*$ by $< *\varphi, f >_x = (f, \varphi)_x$. The vector space H is a Hilbert space with the scalar product $(\oplus f_j, \oplus \varphi_j)_H = \sum_{j=0}^{p-1} \int_S (f_j, \varphi_j)_x ds$. We consider in H the subset H_0 which is formed by elements of the form $\oplus B_j f$ where $f \in S(\overline{D})$. We obtain more than is asserted in the lemma if we prove that H_0 is dense in H. Using the Hahn-Banach theorem it is sufficient to show that if Φ is a continuous linear functional on H which is equal to zero on H_0, $\Phi \equiv 0$. Let Φ be such a functional. According to the theorem of Riesz, there are elements $\widetilde{\varphi}_j \in L^2(G_{j|S})$ $(0 \le j \le p-1)$ such that $\Phi(\oplus f_j) = (\oplus f_j, \oplus \widetilde{\varphi}_j)$ for all $\oplus f_j \in H$. Having extended each of the sections $\widetilde{\varphi}_j$ by zero to $\partial D \backslash S$ we obtain the sections $\varphi \in L^2(G_{j|\partial D})$ $(0 \le j \le p-1)$, and we set $g_j = *\varphi_j$, that is, $g_j \in L^2(G^*_{j|\partial D})$. Since the functional Φ vanishes on H_0, we have $\int_{\partial D} \sum_{j=0}^{p-1} < g_j, B_j f >_x ds = 0$ for all $f \in S(\overline{D})$. We can now use Theorem 29.9 from the book of Tarkhanov [63] and conclude that there exists a section $g \in H^2_{P',C}(D)$ for which $C_j g = g_j$ $(0 \le j \le p-1)$ on ∂D. In particular, $C_j g = 0$ $(0 \le j \le p-1)$ on $\partial D \backslash S$. According to Theorem 2.8, $g \equiv 0$ in D, so that $\Phi \equiv 0$, which was to be proved. □

To apply the results of §1 to Problem 3.1 some information about the orthogonal projection in $L^2(\oplus G_{j|\partial D})$ on the subspace formed by elements of the form $\oplus B_j f$, where $f \in H^2_{P,B}(D)$ is needed. We can obtain it by the very general theory of functional spaces with reproducing kernels (see Aronszajn [4]). We now explain this. We consider the space $H = H^2_{P,B}(D)$ together with the hermitian form

$$(3.1) \qquad (f, v) = \sum_{j=0}^{p-1} \int_{\partial D} (B_j f, B_j \varphi)_x ds \quad (f, \varphi \in H^2_{P,B}(D)$$

on it. Theorem 2.8 implies that any solution $f \in H^2_{P,B}(D)$ is completely defined by the restrictions of the expressions $B_j f$ $(0 \leq j \leq p-1)$ to ∂D. Hence the form (3.1) defines a scalar product on $H^2_{P,B}(D)$.

Lemma 3.3. *$H^2_{P,B}(D)$ is a separable Hilbert space.*

Proof. We can identify the pre-Hilbert space $H^2_{P,B}(D)$ with the subspace of $L^2(\oplus G_{j|\partial D})$ formed by the elements of the form $\oplus B_j f$, where $f \in H^2_{P,B}(D)$. However by Theorem 29.3 of see Tarkhanov [63] one can quite simply notice that this subspace is closed. In fact, it is the intersection of kernels of special continuous linear functionals on $L^2(\oplus G_{j|\partial D})$. Hence, $H^2_{P,B}(D)$ inherits the properties of a closed subset of the separable Hilbert space. This proves the the lemma. □

Let x be a fixed point of the domain D. We consider the functional $\delta_x^{(j)}$ $(1 \leq j \leq k)$ on $H^2_{P,B}(D)$ given by $\delta_x^{(j)} f = f^{(j)}(x)$ $(1 \leq j \leq k)$ where $f^{(j)}(x)$ is the j-th component of f at the point x. Formula (2.1) implies that this functional is continuous on $H^2_{P,B}(D)$. Moreover, a stronger property than continuity holds. Namely, for any compact $K \subset D$ there is a constant C_K such that $\|\delta_x^{(j)}\| < C_K$ for $x \in K$. Hence, $H^2_{P,B}(D)$ is a space with a reproducing kernel (see Aronszajn [4]). We can now use the Riesz theorem on the general form of a continuous linear functional on a Hilbert space and thus find (unique) elements $\mathcal{K}_x^{(j)} \in H^2_{P,B}(D)$ $(1 \leq j \leq k)$ such that $f^{(j)}(x) = (f, \mathcal{K}_x^{(j)})_H$ for all $f \in H$. We denote by $\mathcal{K}_x^{(i,j)}$ $(1 \leq j, i \leq k)$ the i-th component of the vector-valued function $\mathcal{K}_x^{(j)}$. The (well defined) matrix $\mathcal{K}(x,y) = \|\mathcal{K}_x^{(i,j)}(y)\|$ is called the reproducing kernel of the domain D relative to $H^2_{P,B}(D)$. Its properties are well-known.

Proposition 3.4. *The matrix $\mathcal{K}(x,y)$ is hermitian, that is, $\mathcal{K}(x,y)^* = \mathcal{K}(y,x)$.*

Proof. If $1 \leq j, i \leq k$ then

$$\mathcal{K}_y^{(i,j)}(x) = (\mathcal{K}_y^{(j)}, \mathcal{K}_x^{(i)})_H = \overline{(\mathcal{K}_x^{(i)}, \mathcal{K}_y^{(j)})_H} = \overline{\mathcal{K}_x^{(i,j)}(y)},$$

which was to be proved. □

PROPOSITION 3.5. *$tr\mathcal{K}(x,x) = \|\delta_x^{(j)}\|$.*

PROOF. We have,

$$tr\mathcal{K}(x,x) = \sum_{j=1}^{k} (\mathcal{K}_x^{(j)}, \mathcal{K}_x^{(j)})_H = \|\delta_x^{(j)}\|,$$

which was to be proved. □

PROPOSITION 3.6. *If $\{e_\nu\}$ is an orthonormal basis of the space $H^2_{P,B}(D)$ then for all $x \in D$ we have $\mathcal{K}_x^{(j)} = \sum_{\nu=1}^{\infty} \overline{e_\nu^{(j)}(x)} e_\nu$ $(1 \le j \le k)$ where the series converges in the norm of H. As a series of (vector-) functions of two variables $(x,y) \in D \times D$, it converges uniformly on compact subsets of $D \times D$.*

PROOF. For a fixed $x \in D$ the Fourier series of the element $\mathcal{K}_x^{(j)} \in H^2_{P,B}(D)$ $(1 \le j \le k)$ with respect to the basis $\{e_\nu\}$ has the form $\mathcal{K}_x^{(j)} = \sum_{\nu=1}^{\infty} (\mathcal{K}_x^{(j)}, e_\nu)_H e_\nu$. To prove the first part of the proposition we notice that $(\mathcal{K}_x^{(j)}, e_\nu)_H = \overline{e_\nu^{(j)}(x)}$. We suppose now that K_i $(i = 1,2)$ are compact subsets of D, and that constants C_i $(i = 1,2)$ are chosen so that $\|\delta_x^{(j)}\| \le C_i$ for $x \in K_i$. Then for $x \in K_i$

$$\left(\sum_{\nu=1}^{\infty} |\overline{e_\nu^{(j)}(x)}|^2 \right)^2 \le |\sum_{\nu=1}^{\infty} \overline{e_\nu^{(j)}(x)} e_\nu(x)|^2 \le$$

$$\le C_i \| \sum_{\nu=1}^{\infty} \overline{e_\nu^{(j)}(x)} e_\nu(y) \|^2 = C_i \sum_{\nu=1}^{\infty} |e_\nu^{(j)}(x)|^2.$$

Hence here we have $\sum_{\nu=1}^{\infty} |e_\nu^{(j)}(x)|^2 \le C_i$ for $x \in K_i$ $(i = 1,2)$. Thus, if $(x,y) \in K_1 \times K_2$, we obtain

$$\sum_{\nu=1}^{\infty} |\overline{e_\nu^{(j)}(x)} e_\nu(y)| \le \left(\sum_{\nu=1}^{\infty} |e_\nu^{(j)}(x)|^2 \right)^{1/2} \left(\sum_{\nu=1}^{\infty} |e_\nu(y)|^2 \right)^{1/2} \le \sqrt{kC_1C_2}.$$

This proves the absolute and uniform convergence on compact subsets of $D \times D$ of the series for $\mathcal{K}_x^{(j)}$, which was to be proved. □

The formula for the reproducing kernel mentioned in Proposition 3.6 could be written in the form $\mathcal{K}(x,y) = \sum_{\nu=1}^{\infty} e_\nu(x)^* \otimes e_\nu(y)$. The a priori estimations for a solution of an elliptic system imply that this series here converges uniformly together with all its derivatives on compact subsets of $D \times D$, that is, $\mathcal{K}$ is an infinitely differentiable section of $E \boxtimes E$ over $D \times D$.

THEOREM 3.7. *For all solutions $f \in H^2_{P,B}(D)$ the following formula holds*

$$f(x) = \int_{\partial D} \sum_{j=0}^{p-1} < *B_j \mathcal{K}(x,.), B_j f >_y ds \quad (x \in D). \tag{3.2}$$

PROOF. We simply rewrite the reproducing property of the kernel $\mathcal{K}$ in detail. □

For holomorphic functions of several variables Theorem 3.7 is due to Bungart [7].

COROLLARY 3.8. *In the space $L^2(\oplus G_{j|\partial D})$ the operator of the orthogonal projection on the subspace Σ_1 formed by elements of the form $\oplus B_j f$ where $f \in H^2_{P,B}(D)$, has the form*

$$\Pi(\oplus f_j) = \oplus B_j \left(\int_{\partial D} \sum_{i=0}^{p-1} < *B_i \mathcal{K}(x,.), f_i >_y ds \right) \quad (\oplus f_j \in L^2(\oplus G_{j|\partial D}). \tag{3.3}$$

PROOF. Let $\{e_\nu\}$ be an orthonormal basis of the space $H^2_{P,B}(D)$. Then, from equality (3.1), $\{\oplus B_j e_\nu\}$ is an orthonormal basis of the subspace Σ_1 in $L^2(\oplus G_{j|\partial D})$. Hence if $\oplus f_j \in L^2(\oplus G_{j|\partial D})$ then

$$\Pi(\oplus f_j) = \sum_{\nu=1}^{\infty} (\oplus f_j, \oplus B_j e_\nu)_{L^2(\oplus G_{j|\partial D})} (\oplus B_j e_\nu) =$$

$$= \oplus B_j \left(\sum_{\nu=1}^{\infty} (\oplus f_j(y), \oplus B_j(y)(e^*_\nu(x) \otimes e_\nu(y)))_{L^2(\oplus G_{j|\partial D})} (\oplus B_j e_\nu) \right).$$

The first part of Proposition 3.6 implies that the sign of summation over ν can be taken inside sign of the scalar product. This gives at once formula (3.3), which was to be proved. □

We outline a scheme of application of the theory of §1 to the Cauchy Problem 3.1. We set $H_1 = L^2(\oplus G_{j|\partial D})$ and $H_2 = L^2(\oplus G_{j|S})$. The hermitian structures on these spaces are introduced as was explained in the proof of Lemma 3.2. Then H_1 and H_2 are Hilbert spaces. The operator $T : H_1 \to H_2$ is given by the restrictions of sections. Then the adjoint operator T^* is simply the extension of sections from S to $\partial D \backslash S$ by zero. Further, we consider in H_1 the subspace Σ_1 formed by elements of the form $\oplus B_j f$ where $f \in H^2_{P,B}(D)$. We have already noted that Σ_1 is a closed subspace of H_1 representing $H^2_{P,B}(D)$. We denote by Π the operator of orthogonal projection on Σ_1 in H_1. This is the integral operator given by formula (3.3). Lemma 3.2 means that the operator $T : \Sigma_1 \to H_2$ has a dense image, therefore we set $\Sigma_2 = H_2$. We must consider the mapping $\Pi T^*T : \Sigma_1 \to \Sigma_1$, which is given by the integral (3.3) except that the domain of integration is S instead of ∂D. If the set S has at least one interior point (on ∂D) then, from Theorem 2.8, the operators $T : \Sigma_1 \to \Sigma_2$ and $\Pi T^*T : \Sigma_1 \to \Sigma_1$ are injective. Even in the simplest situations the operator ΠT^*T is not compact, moreover, it is not Carleman operator (see Berezanskii [5], ch.V, 14). Let $\{b_\lambda^{(i)}\}$ be a complete system of generalized eigen vectors of the operator ΠT^*T in L_1' where $L \subset \Sigma_1 \subset L_1'$ is a suitable equipment of Σ_1. Then Corollaries 1.7 and 1.8 imply the following results.

Theorem 3.9. *We assume that the complement of S in ∂D has at least one interior point. Then for the solvability of Problem 3.1 it is necessary and sufficient that*

$$\int_{-0}^{1} \sum_{i=1}^{N_\lambda} |\frac{(\Pi T^*(\oplus f_j), b_\lambda^{(i)})_{H_1}}{\lambda}|^2 d\sigma(\lambda) < \infty \tag{3.4}$$

Proof. It is sufficient to note that in this case we have $m = \|T\|^2 = 1$. □

It is clear that Theorem 3.9 has only theoretical value, but is not in the least a practical, because its application depends on the singular eigenvalue problem for the operator ΠT^*T. Therefore cases where one succeeds in calculating the system $\{b_\lambda^{(i)}\}$ in an explicit form are very interesting. There is such a situation in one of the simplest Cauchy problems for holomorphic functions, considered by Krein and Nudelman [26] (see Example 1.10). A corresponding result holds for Carleman s formula.

Theorem 3.10. *Let $\partial D \backslash S$ have a non-empty interior (in ∂D). Then under condition (3.4) the solution of Problem 3.1 is given by the formula*

$$f(x) = -\int_{-0}^{1} (*^{-1} \sum_{i=1}^{N_\lambda} (\oplus C_j \Phi(x,.)), b_\lambda^{(i)})_{H_1} \frac{(\Pi T^*(\oplus f_j), b_\lambda^{(i)})_{H_1}}{\lambda} d\sigma(\lambda) \tag{3.5}$$

Proof. It is sufficient to substitute the expressions $\oplus B_j f(y)$ $(y \in D)$, obtained by Corollary 1.8, in Green formula (2.1). □

A similar formula could be constructed on the basis of the integral representation (3.2). æ

§4. Weak values of solutions in $L^q(D)$ on the boundary of D

Again let P be a differential operator with an injective symbol on X, not necessarily satisfying the condition $(U)_S$, and f be a solution of the system $Pf = 0$ in D of Lebesgue class $L^q(E_{|D})$ where $1 \le q \le \infty$. What can one say of the limit values on ∂D of the expressions $B_j f$ $(0 \le j \le p-1)$? Extrapolating the situation for holomorphic functions one can say that the class of solutions $S(D) \cap L^q(E_{|D})$ is wider than $H^2_{P,B}(D)$. Moreover, à priori it is not clear, whether the solution $f \in S(D) \cap L^q(E_{|D})$ has finite order of growth near ∂D, that is whether the expressions $B_j f$ $(0 \le j \le p-1)$ have weak limit values on ∂D. Estimates of growth near ∂D of solutions $f \in S(D) \cap L^2(E_{|D})$ could be obtained from the asymptotic behaviour of the reproducing kernel of the domain D with respect to the Hilbert space $S(D) \cap L^2(E_{|D})$. However even in the case of the Cauchy-Riemann system this asymptotic behaviour is not known for all domains (see Henkin [15], p.68). In this section we prove that for any solution $S(D) \cap L^1(E_{|D})$ there are weak limit values of the expressions $B_j f$ $(0 \le j \le p-1)$ on the boundary. Then the theorem of Rojtberg [47] allows us to know the smoothness of these values on ∂D.

So, we fix $f \in S(D) \cap L^q(E_{|D})$, where $1 \le q \le \infty$, and a number j $(0 \le j \le p-1)$. Putting aside for the meanwhile the questions of the correctness of the definition, we associate a vector-valued distribution $f_j \in \mathcal{D}'(G_{j|\partial D})$ with the solution f in the following way. Let $g_j \in C^{b_j+1}(G^*_{j|\partial D})$. Using Lemma 28.2 of Tarkhanov [63], we find a section $g \in C^p_{loc}(F^*)$ such that $C_j g = g_j$, and $C_i g = 0$ for $i \ne j$ on ∂D. Then we set

$$< g_j, f_j >= - \int_D < P' g, f >_x dv \quad (g_j \in C^{b_j+1}(G_{j\partial D}) \tag{4.1}$$

Lemma 4.1. *Definition (4.1) is correct, that is, it does not depend on the choice of the section $g \in C^p_{loc}(F^*)$ for which $C_j g = g_j$, and $C_i g = 0$ for $i \ne j$ on ∂D.*

Proof. It is sufficient to show that, if for a section $g \in C^p_{loc}(F^*)$ the boundary values on ∂D of the expressions $C_j g$ $(0 \le j \le p-1)$ are equal to zero, then $\int_D < P' g, f > dv = 0$.

First of all we replace the section g by another section with the same differential $P' g$, and with derivatives up to order $(p-1)$ are equal to zero on ∂D. For this we represent the section g in D by means of the homotopy formula on a manifold with boundary (see, for example, Tarkhanov [63], (12.3)). Bearing in mind the connection between the Green operators of the differential operator P and the transposed of P, and using Lemma 2.3 we have

$$\Phi'(\chi_D P' g) + P^{1'} \Phi'(\chi_D g) + \mathcal{S}^{1'}(\chi_D g) = \chi_D g. \tag{4.2}$$

Let $v \in W^{2p,\widetilde{q}}(E^{2^*})$ (where $\widetilde{q} >> 1$) be an extension of the section $\Phi(\chi_D g)$ from $X \backslash D$ to the whole set X. The number $\widetilde{q}$ can be chosen as large as we want, however for our purposes it is sufficient that $\widetilde{q} > n$, and $\widetilde{q} \ge q'$ where q' is dual to the index q, that is, $1/q + 1/q' = 1$. Then, if we consider the section $\widetilde{g} = \Phi(\chi_D P' g) + P^{1'} v + S^{1'}(\chi_D g)$, we can say that $g \in W^{p,\widetilde{q}}(F^*)$, and $P' \widetilde{g} = P' g$. Moreover, from formula (4.2), $\widetilde{g} \equiv 0$ outside of D, but since $\widetilde{g} \in C^{p-1}_{loc}(F^*)$ we have $D^\alpha \widetilde{g} = 0$ $(|\alpha| \le p-1)$ on ∂D. Then, replacing if necessary g by $\widetilde{g}$, we assume without loss of generality that the derivatives of g up to order $(p-1)$ vanish on ∂D. In this case there is some loss of smoothness of g, but this is not important for us. Further, we use the lemma of Bochner which says that for any $\varepsilon > 0$ there is a function $\varphi_\varepsilon \in \mathcal{D}(X)$ $(0 \leq \varphi_\varepsilon \leq 1)$ with support in the ε-neighbourhood of the boundary ∂D which is equal to unit in some smaller neighborhood of ∂D, and for which $|D^\alpha \varphi_\varepsilon| \leq c_\alpha \varepsilon^{-|\alpha|}$ everywhere in $\mathbb{R}^n$ where the constant c_α does not depend on ε (see Hörmander [16], theorem 1.4.1). We have

$$\int_D < P'g, f >_x dv = \int_D < P'(1-\varphi_\varepsilon)g, f >_x dv + \int_D < P'(\varphi_e g), f >_x dv \tag{4.3}$$

Since the section $(1-\varphi_\varepsilon)$ has compact support in D then, from Stokes' formula, the first summand on the right hand side of (4.3) disappears. As for the second summand we can write

$$\int_D < P'(\varphi_\varepsilon g), f >_x dv = \sum_{|\alpha|\leq p} (-1)^{|\alpha|} \sum_{|\beta|\leq|\alpha|} \binom{\alpha}{\beta} \int_{D\setminus D_\varepsilon} < D^\beta \varphi_\varepsilon D^{\alpha-\beta}(P^T_\alpha g), f >_x dv. \tag{4.4}$$

We want to prove that the right hand side converges to zero, as $\varepsilon \to +0$. For to do this it is sufficient to estimate the typical summand in (4.4): $\int_{D\setminus D_\varepsilon} < D^\beta \varphi_\varepsilon D^{\alpha-\beta}(P^T_\alpha g), f >_x dv$ $(\beta \neq 0)$. Having used the Hölder inequality, and taking into consideration the estimates of the derivatives of the function φ_ε we obtain with a constant $c > 0$ which does not depend on ε such that

$$\left| \int_{D\setminus D_\varepsilon} < D^\beta \varphi_\varepsilon D^{\alpha-\beta}(P^T_\alpha g), f >_x dv \right| \leq$$

$$\leq \|D^\beta \varphi_\varepsilon D^{\alpha-\beta}(P^T_\alpha g)\|_{L^{q'}(F^*_{|D\setminus D_\varepsilon})} \|f\|_{L^q(E_{D\setminus D_\varepsilon})} \leq$$

$$\leq c_1 \varepsilon^{-|\beta|} \|D^{\alpha-\beta} g\|_{L^{q'}(F^*_{|D\setminus D_\varepsilon})} \|f\|_{L^q(E_{D\setminus D_\varepsilon})} \tag{4.5}$$

Since $g \in C^{p-1}_{loc}(F^*)$, and $D^\gamma g = 0$ $(|\gamma| \leq p-1)$ on ∂D, using the localization process and the repeated use of the Newton-Leibniz formula, it is not difficult to see there is a constant $c_2 > 0$ such that for all sufficiently small $\delta > 0$ we have

$$\|D^{\alpha-\beta} g\|_{L^{q'}(F^*_{|\partial D_\delta})} \leq c_2 \delta^{p-1-|\alpha|+|\beta|+1/q} \|g\|_{W^{p,q'}(F^*_{|D\setminus D_\delta})} \tag{4.6}$$

Similar considerations can be found in the book of Mihailov [39] (p.148). Now we choose $\varepsilon > 0$ sufficiently small and integrate inequality (4.6) with respect to δ from 0 to ε. Then using the Fubini theorem we obtain the inequality

$$\|D^{\alpha-\beta} g\|_{L^{q'}(F^*_{|D\setminus D_\varepsilon})} \leq c_2' \varepsilon^{p-|\alpha|+|\beta|+1/q} \|g\|_{W^{p,q'}(F^*_{|D\setminus D_\varepsilon})}$$

where $c_2' = c_2/((p-1-|\alpha|+|\beta|+1/q)q'+1)^{1/q'}$. Substituting this estimate in (4.5), we obtain

$$|\int_{D\setminus D_\varepsilon} < D^\beta \varphi_\varepsilon D^{\alpha-\beta}(P_\alpha^T g), f >_x dv| \le$$
$$\le c_1 c_2' \varepsilon^{p-|\alpha|+1/q} \|g\|_{W^{p,q'}(F^*_{|D\setminus D_\varepsilon})} \|f\|_{L^q(E_{D\setminus D_\varepsilon})},$$

So we can find a constant $c > 0$ depending only on the norms of the coefficients of the differential operator P in the domain D such that for all sufficiently small $\varepsilon > 0$ we have

$$\left|\int_D < P'g, f >_x dv\right| \le c\|g\|_{W^{p,q'}(E^*_{|D\setminus D_\varepsilon})} \|f\|_{L^q(E_{D\setminus D_\varepsilon})} \tag{4.7}$$

The property of the absolute continuity of a Lebesgue integral with respect to a domain of integration implies that for any q in the range $1 \le q \le \infty$ the expression on the right hand side of (4.7) converges to zero as $\varepsilon \to +0$. Therefore $\int_D < P'g, f >_x dv = 0$, which proves the lemma. □

As one can see, if $q = 1$ in the proof of Lemma 4.1 the arguments fail. Thus in this case the definition (4.1) needs some modification. Namely, it is necessary to change the smoothness of the sections g_j in (4.1) by "+0", that is, we must take, for example, $g \in C^{b_j+1,\lambda}(G^*_{j|\partial D})$, where $\lambda > 0$. The distributions $f_j \in \mathcal{D}'(G_{j|\partial D})$ $(0 \le j \le p-1)$ constructed in (4.1) we now take as the weak limit values of the expressions $B_j f$ on ∂D. It is clear that if $f \in C^{p-1}(E_{|\overline{D}})$ then f_j is simply the pointwise restriction of $B_j f$ on ∂D. However in the general case the identification of f_j $(0 \le j \le p-1)$ with the weak limit values of the expressions $B_j f$ $(0 \le j \le p-1)$ on ∂D by definition (4.1) is difficult. Later on we shall show that this identification is valid, but now we begin with the justification of the naturality of definition (4.1).

LEMMA 4.2. *For any solution $f \in S(D) \cap L^q(E_{|D})$ $(1 < q \le \infty)$ the following Green formula holds:*

$$\int_{\partial D} \sum_{j=0}^{p-1} < C_j g, B_j f >_x ds = -\int_D < P'g, f >_x dv \quad (g \in C^p(F^*_{|\overline{D}})). \tag{4.8}$$

PROOF. For each number $1 \le j \le p-1$ we construct a section $g^{(j)} \in C^p_{loc}(F^*)$ such that $C_j g^{(j)} = C_j g$, and $C_i g^{(j)} = 0$ for $i \ne j$ on ∂D. We set $g_0 = g - g^{(1)} - \ldots - g^{(p-1)}$. Then $g_0 \in C^p_{loc}(F^*_{\overline{D}})$, $C_0 g^{(0)} = C_0 g$, and $C_i g^{(0)} = 0$ for $i \ne 0$ on ∂D. Hence, according to definition (4.1) we can write

$$\int_{\partial D} \sum_{j=0}^{p-1} < C_j g, B_j f >_x ds = \sum_{j=0}^{p-1} \left(-\int_D < P'g^{(j)}, f >_x dv \right) =$$

$$= -\int_D < P'g, f >_x dv,$$

which was to be proved. □

Formula (4.8) holds also for solutions $f \in S(D) \cap L^1(E_{|D})$, however with sections g whose smoothness is greater than "+0", that is, for $g \in C^{p,\lambda}(F^*)$ where $\lambda > 0$.

LEMMA 4.3. *For any solution $f \in S(D) \cap L^1(E_{|D})$ the Green formula (2.1) holds.*

PROOF. Let x be a fixed point belonging to $X \backslash \partial D$. We take some function $\varphi \in \mathcal{D}(X)$ which is equal to 1 in a neighbourhood of ∂D, and vanishes on some neighborhood of the point x. It is clear that $\varphi\Phi \in C^\infty_{loc}(E_x \otimes F^*)$, therefore formula (4.8) implies that

$$\int_{\partial D} \sum_{j=0}^{p-1} < C_j\Phi, B_j f >_x ds = -\int_D < P'(\varphi\Phi, f >_x dv. \tag{4.9}$$

We choose $\varepsilon > 0$ so small that $\varphi \equiv 1$ in some neighbourhood of "the piece" $D \backslash D_\varepsilon$. Since $P'\Phi(x, .) = 0$ everywhere outside of the point x, it follows that the integral on the right hand side of formula (4.9) is equal to the similar integral taken over the domain D_ε. But $f \in S(\overline{D}_\varepsilon)$, therefore the last integral is equal to $-\int_{\partial D_\varepsilon} G_P(\Phi(x, .), f)$, that is, $(\chi_D f)(x)$, which was to be proved. □

We can now formulate the principal result of this section. As before, we denote by $B^{s,q}(G_{j|\partial D})$ the usual Besov spaces of sections of the bundles G_j over ∂D (see Kudrjavtsev and Nikolskii [27]). In particular, if s is not an integer or $q = 2$ then $B^{s,q}(G_{j|\partial D}) = W^{s,q}(G_{j|\partial D})$. If $1 < q < \infty$ then in definition (4.1) we can take $g_j \in B^{b_j+1/q',q}(G_{j|\partial D})$ $(0 \le j \le p-1)$. Lemma 2.2 from the paper of Rojtberg [47] guarantees existence of a section $g \in W^{s,q}(F^*_{|\partial D})$ such that $C_j g = g_j$, and $C_i g = 0$ for $i \ne j$ on ∂D. Then one can substitute g into the right part of (4.1). Moreover, the above-mentioned lemma of Rojtberg [47] says that the mapping $g_j \to g$ is continuous. Using Holder's inequality it is easy to conclude that $B_j f \in B^{-b_j-1/q',q'}(G_{j|\partial D})$ $(0 \le j \le p-1)$ (see our paper [51]). However we obtain a more general result directly from the fundamental theorem of Rojtberg [47].

THEOREM 4.4. *For a solution $f \in S(D) \cap L^1(E_{|D})$ the limit values of the expressions $B_j f$ $(0 \le j \le p-1)$ on ∂D defined by formula (4.1) are the weak limit values. Moreover $f \in W^{s,q}(E_{|D})$ $(1 < q < \infty)$ if and only if $B_j f \in B^{s-b_j-1/q,q}(G_{j|\partial D})$ $(0 \le j \le p-1)$.*

PROOF. Again we shall try to reduce the proof to the corresponding fact for solutions of elliptic systems. We fix a section $f \in S(D) \cap L^q(E_{|D})$, $q > 1$, satisfying $Pf = 0$ in D. Then f must also satisfy $\Delta f = 0$ where $\Delta = P^*P$ is an elliptic differential operator of type $E \to E$, and of order $2p$ on X. The system $\{B_j\}_{j=0}^{p-1}$

can be replaced with a Dirichlet system of order $(2p-1)$ on ∂D in the following way. We set $\widetilde{B_j} = B_j$ for $0 \le j \le p-1$, and $\widetilde{B_j} = *^{-1} C_{j-p} * P$ for $p \le j \le 2p-1$. Then $\{\widetilde{B_j}\}_{j=0}^{2p-1}$ is a Dirichlet system of order $(2p-1)$ on ∂D, and the Dirichlet system $\{\widetilde{C_j}\}_{j=0}^{2p-1}$ corresponding to it by Lemma 2.3 (with $P = \Delta$) has the form $\widetilde{C_j} = -C_j * P *^{-1}$ for $0 \le j \le p-1$, and $\widetilde{C_j} = - * B_{j-p} *^{-1}$ for $p \le j \le 2p-1$. We now use a relation (which is similar to (4.1) to define the limit values of the expressions $\widetilde{B_j} f$ $(0 \le j \le 2p-1)$ on ∂D in our new situation. More precisely, these expressions are only interesting for $(0 \le j \le p-1)$. So, let $g \in C^{b_j+1}(G^*_{j|\partial D})$ $(0 \le j \le p-1)$. Using Lemma 28.2 of Tarkhanov [63] we find a section $\mathcal{G} \in C^{2p}_{loc}(E^*)$ such that $C_j * P *^{-1} \mathcal{G} = g$, and $\widetilde{C_i}\mathcal{G} = 0$ for $i \ne j$ $(0 \le i \le 2p-1)$ on ∂D. Then we set

$$(4.10) \qquad < g, B_j f > = - \int_D < \Delta' \mathcal{G}, f >_x dv, \quad (g_j \in C^{b_j+1}(G^*_{j|\partial D})).$$

However, if we define $B_j f$ on ∂D by means of formula (4.1), the choice of g in Lemma 4.1 is unimportant. In particular, nothing prevents us from taking $g = *P *^{-1} \mathcal{G}$ in (4.1). Then we obtain equality (4.10). Hence the definition of the limit values of $B_j f$ $(0 \le j \le p-1)$ on ∂D does not depend on whether f is a solution of the system $Pf = 0$ or $\Delta f = 0$. So, replacing the operator P by Δ we may suppose without loss of a generality that P is elliptic. But then the first part of Theorem 4.4 follows from Lemmata 4.3 and 2.7. For, from Lemma 4.3, the solution f is represented by the limit values of the expressions $B_j f$ $(0 \le j \le p-1)$ on ∂D which are defined in accordance with equality (4.1) by means of the Green formula (2.1). And Lemma 2.7 asserts that the weak jump in going across ∂D of the expressions $B_j \mathcal{G}(\oplus B_i f)$ $(0 \le j \le p-1)$ coincides with $B_j f$. Hence the limit values of the expressions $B_j f$ $(0 \le j \le p-1)$ on ∂D exist, and they coincide with the limit values calculated by the formula (4.1). This proves the first part of the theorem for solutions $f \in L^q(E_{|D})$ $(q > 1)$, and for $q = 1$ we must make obvious modifications. To prove the second part of the theorem we assume in addition that $f \in S(D) \cap W^{s,q}(E_{|D})$ where $1 < q < \infty$. Rojtberg [47] proved that there are limit values of the expressions $B_j f$ $(0 \le j \le p-1)$ on ∂D in the following sense. There is a sequence $f^{(\nu)} \in C^\infty(E_{|\overline{D}})$ such that $f^{(\nu)}$ converges to f in $W^{s,q}(E_{|D})$ and Pf converges to zero in $W^{s-p,q}(F_{|D})$. Moreover, for any such a sequence $f^{(\nu)}$ the sequence $B_j f^{(\nu)}$ $(0 \le j \le p-1)$ is fundamental in Besov space $B^{s-b_j-1/q,q}(G_{j|\partial D})$, and therefore it converges in this space to a limit f_j. Arguing as in the proof of Theorem 2.6 we see that the solution f is represented by the boundary values f_j by means of the Green formula (2.3). Then Lemma 2.7 again shows that the sections f_j $(0 \le j \le p-1)$ are the limit values on ∂D of the expressions $B_j f$. So the weak limit values of the expression $B_j f$ $(0 \le j \le p-1)$ on ∂D belong to the Besov space $B^{s-b_j-1/q,q}(G_{j|\partial D})$.

Conversely, if such an inclusion holds then formula (2.1) and the theorems on boundedness of potential (or co-boundary) operators on a manifold with boundary (see Rempel and Schulze [45], 2.3.2.5) imply that $f \in W^{s,q}(E_{|D})$. This proves Theorem 4.4. □

This theorem, in particular, shows that for a solution $f \in S(D) \cap L^1(E_{|D})$ definition (4.1) of the boundary values $B_j f$ $(0 \le j \le p-1)$ on ∂D does not depend on the choice of the differential operator P. æ

§5. The Green integral and solvability of the Cauchy problem for elliptic systems

In this and the following 3 sections we assume that P is an elliptic differential operator such that the transposed operator P' satisfies the uniqueness condition of the Cauchy problem in the small on X.

Theorem 4.4 explains that if we solve Problem 2.1 (of Cauchy) in the class $S(D) \cap L^q(E_{|D})$ (or, more generally, in the class of sections satisfying $Pf = 0$ in D which have finite order of growth near the boundary of D) then we can hope only for generalized limit values of the expressions $B_j f$ $(0 \le j \le p-1)$ on ∂D. Therefore, since distributions have restrictions only on open subsets of the domain, it is natural to assume that S is an open connected piece (subdomain) of the boundary of D.

This situation can be realized in the following way. There is some domain $O \Subset X$, and S is a smooth closed hypersurface in O dividing this domain into two connected components: $O^- = D$ and $O^+ = O \backslash \overline{D}$.

In the wording of the following problem there are Besov spaces $B^{s-b_j-1/q,q}(G_{j|\overline{S}})$ whose definition may be not clear. We define these spaces in the following way. In Besov space $B^{s-b_j-1/q,q}(G_{j|\partial D})$ (defined by one of the usual method) we consider the subspace Σ formed by all the sections which are equal to zero on $\overline{S}$. For $s < 0$ this means that $< g, f >= 0$ for all $g \in B^{-s,q'}(G^*_{j|\partial D})$ with $supp\, g \subset \overline{S}$. It is easy to see that Σ is closed. The corresponding quotient space (with the quotient topology) we denote by $B^{s-b_j-1/q,q}(G_{j|\overline{S}})$

Problem 5.1. *Let $f_j \in B^{s-b_j-1/q,q}(G_{j|\overline{S}})$ $(0 \le j \le p-1)$ be known sections on S where $s \in \mathbb{Z}_+$, and $1 < q < \infty$. It is required to find a section $f \in S(D) \cap W^{s,q}(E_{|D})$ such that $B_j f = f_j$ $(0 \le j \le p-1)$ on S.*

Under the formulated conditions the operator P has a right fundamental solution on X. In other words there is an operator $\Phi \in pdo_{-p}(F \to E)$ such that $\Phi P = 1 - \mathcal{S}^0$ on $C^\infty_{comp}(E)$ where $\mathcal{S}^0 \in pdo_{-\infty}(E \to E)$ is some smoothing operator. Then $P S^0 = 0$ on generalized sections of E with compact supports (that is, on $\mathcal{E}'(E)$).

Using the "initial" data of Problem 5.1 we construct the Green integral in a the special way. That is, we denote by $\widetilde{f}_j \in B^{s-b_j-1/q,q}(G_{j|\partial D})$ $(0 \le j \le p-1)$ an extension of the section f_j to the whole boundary. If, for example, $s = 0$ and $f_j \in L^2(G_{j|S})$ $(0 \le j \le p-1)$, then it is possible to extend them by zero on $\partial D \backslash S$. In any case the extensions could be chosen so that they will be supported on a given neighbourhood of the compact $\overline{S}$ on ∂D. Then we set $\widetilde{f} = \oplus \widetilde{f}_j$, and

$$\mathcal{G}(\widetilde{f})(x) = \int_{\partial D} < C_j \Phi(x,.), \widetilde{f}_j >_y ds \quad (x \neq\in \partial D) \tag{5.1}$$

It is clear that $\mathcal{G}(\widetilde{f})$ is a solution of the system $Pf = 0$ everywhere in $X \backslash \partial D$. In particular, if we denote by $\mathcal{F}^{\pm}$ the restrictions of a section $\mathcal{F} \in D'(E_{|O})$ to the sets $O^{\pm}$, then $\mathcal{G}(\widetilde{f})^{\pm} \in S(O^{\pm})$.

Theorem 5.2. *If the boundary of the domain D is sufficiently smooth then, for Problem 10.1 to be solvable, it is necessary and sufficient that the integral $\mathcal{G}(\widetilde{f})$ extends from O^+ to the whole domain O as a solution belonging to $S(O) \cap W^{s,q}(E_{|O})$.*

Proof. Necessity. Suppose that there is a section $f \in S(D) \cap W^{s,q}(E_{|D})$ such that $B_j f = f_j$ $(0 \leq j \leq p-1)$ on S.

We consider the following section in the domain O (more exactly, in $O \backslash S$):

$$\mathcal{F}(x) = \begin{cases} \mathcal{G}\widetilde{f}(x), x \in O^+, \\ \mathcal{G}\widetilde{f}(x) + f(x), x \in O^-. \end{cases} \tag{10.2}$$

Using the boundedness theorem for potential operators in Sobolev spaces on manifolds with boundary (see Rempel and Schulze [45], 2.3.2.5) we can conclude that $\mathcal{G}(\widetilde{f})^{\pm} \in W^{s,q}(E_{|O^{\pm}})$ (differentiability $max(s, p-s)$ is sufficient). This means $\mathcal{F}^{\pm} \in W^{s,q}(E_{|O^{\pm}})$.

On the other hand, we consider the difference $\Delta = \mathcal{G}(\oplus B_j f) - \mathcal{G}(\widetilde{f})$. Let $\varphi_\varepsilon \in D(X)$ be any function supported on the ε-neighbourhood of the set $\partial D \backslash S$, and equal to 1 in some smaller neighbourhood of this set. Since $B_j f = \widetilde{f}_j$ $(0 \leq j \leq p-1)$ on S then we can write

$$\Delta(x) = \int_{\partial D} \sum_{j=0}^{p-1} < C_j \Phi(x,.), \varphi_\varepsilon (B_j f - \widetilde{f}_j) >_y ds \quad (x \notin \partial D).$$

The right hand side of this equality is a solution of the system $Pf = 0$ everywhere in the domain O except the part of the ε-neighbourhood of the boundary of S on ∂D which belongs to O. Therefore, since $\varepsilon > 0$ is arbitrary, $\Delta \in S_P(O)$.

Now using the expression for the integral $\mathcal{G}(\oplus B_j f)$ from the Green formula (2.3) and puting $\mathcal{G}(\widetilde{f}) = \mathcal{G}(\oplus B_j \widetilde{f}) - \Delta$ in inequality (5.2) we obtain

$$\mathcal{F}(x) = -\Delta(x) \quad (x \in O \backslash S)$$

Since $\mathcal{S}^0(\chi_D f) \in S(X)$ the section $\mathcal{F}$ extends to the whole domain O as a solution of the system $Pf = 0$.

Hence the section $\mathcal{F}$ extends to the whole domain O as a solution of the system $Pf = 0$.

Thus, $\mathcal{F}$ belongs to $S(O) \cap W^{s,q}(E_{|O})$, and on O^+ this section coincides with $\mathcal{G}(\widetilde{f})^+$, which was to be proved.

Sufficiency. Conversely, let $\mathcal{F} \in S(O) \cap W^{s,q}(E_{|O})$ be a solution coinciding with $\mathcal{G}(\widetilde{f})^+$ on O^+. We set $f(x) = -\mathcal{G}(\widetilde{f}) + \mathcal{F}(x)$ $(x \in D)$. The above mentioned boundedness theorem for potential operators in Sobolev spaces (see Rempel and Schulze [45], 2.3.2.5) implies that $\mathcal{G}(\widetilde{f}) \in W^{s,q}(E_{|O^-})$. Therefore $f \in S(D) \cap W^{s,q}(E_{|D})$.

Now, for $g_j \in \mathcal{D}(G^*_{j|S})$ $(0 \le j \le p-1)$, Lemma 2.7 implies that

$$\lim_{\varepsilon \to +0} \int_{\partial D} < g, B_j f(x - \varepsilon\nu(x)) >_x ds = \lim_{\varepsilon \to +0} \int_S < g, B_j f(x - \varepsilon\nu(x)) >_x ds =$$

$$= \lim_{\varepsilon \to +0} \int_S < g, -B_j(\mathcal{G}(\widetilde{f}))(x - \varepsilon\nu(x)) + B_j\mathcal{F}(x - \varepsilon\nu(x)) >_x ds =$$

$$= \lim_{\varepsilon \to +0} \int_S < g, -B_j(\mathcal{G}(\widetilde{f}))(x - \varepsilon\nu(x)) + B_j\mathcal{F}(x + \varepsilon\nu(x)) >_x ds =$$

$$= \lim_{\varepsilon \to +0} \int_S < g, -B_j(\mathcal{G}(\widetilde{f}))(x - \varepsilon\nu(x)) + B_j(\mathcal{G}(\widetilde{f}))(x + \varepsilon\nu(x)) >_x ds =$$

$$= \int_S < g_j, \widetilde{f}_j >_x ds = \int_S < g_j, f_j >_x ds.$$

Hence $B_j f = f_j$ $(0 \le j \le p-1)$ on S, that is, f is a soution of Problem 5.1, which was to be proved. □

æ

§6. A solvability criterion for the Cauchy problem for elliptic systems in the language of space bases with double orthogonality

Theorem 5.2 has been formulated so that the application of the theory of §1 (see part 1) is suggested. For this assume in addition that $q = 2$.

So, in this section we consider the solvability aspect of Problem 5.1.

Problem 6.1. *Under what conditions on the sections* $f_j \in W^{s-b_j-1/2,2}(G_{j|\overline{S}})$ $(0 \le j \le p-1)$ *is there a solution* $f \in S(D) \cap W^{s,2}(E_{|D})$ *such that* $B_j f = f_j$ $(0 \le j \le p-1)$ *on* S *?*

Let Ω be some relatively compact subdomain of O^+. Since $\Omega \Subset O^+$, it follows that the restriction to Ω of the Green integral $\mathcal{G}(\widetilde{f})$ defined by equality (5.1) belongs to the space $S(\Omega) \cap W^{s,2}(E_{|\Omega})$. Hence the extendibility condition for $\mathcal{G}(\widetilde{f})$ from O^+ to the whole domain O (as a solution in the class $S(O) \cap W^{s,2}(E_{|O})$ could be obtained by the use of a suitable system $\{b_\nu\}$ in $S(O) \cap W^{s,2}(E_{|O})$ with the double orthogonality property. More exactly, it is required that $\{b_\nu\}$ should be an orthonormal basis in $\Sigma_1 = S(O) \cap W^{s,2}(E_{|O})$ and an orthogonal basis in $\Sigma_2 = S(\Omega) \cap W^{s,2}(E_{|\Omega})$ (or the contrary !).

How can such a system be constructed ? The theory of §1 answers this question.

We consider Sobolev spaces $H_1 = W^{s,2}(E_{|O})$ and $H_2 = W^{s,2}(E_{|\Omega})$ of sections of E. According to our approach we define them in the "interior" way using the Riemannian metric dx on O or Ω, and the Hermitian metric on (fibers of) E. Thus, H_1 and H_2 are Hilbert spaces. On the other hand, if the boundaries of O and W satisfy minimal conditions of the smoothness (roughly speaking they should be Lipschitz's ones) then these spaces are isomorphic (as normed spaces) to the Hilbert spaces $W^{s,2}(E_{|\overline{O}})$ and $W^{s,2}(E_{|\overline{\Omega}})$. These spaces are already defined in the "exterior" way. Namely, they are defined as quotient spaces of the Hilbert space $W^{s,2}(E)$ by closed subspaces of sections vanishing on $\overline{O}$ or $\overline{\Omega}$ respectively.

The operator $T : H_1 \to H_2$ is given by restriction of sections so that this is a continuous linear mapping of the Hilbert spaces.

Further, we distinguish in H_1 and H_2 the subspaces Σ_1 and Σ_2 which are formed by sections $\mathcal{F}$ satisfying $P\mathcal{F} = 0$ in O or Ω respectively. The Stiltjes-Vitali theorem (see Hormander [16],4.4.2) implies that these subspaces are closed, therefore they are Hilbert spaces with the induced hermitian structures.

It is clear that the restriction of the mapping T to Σ_1 maps to Σ_2. However it is not evident that the image of T is dense in Σ_2.

LEMMA 6.2. *If the boundary of the domain $\Omega \Subset O$ is regular, and the complement of Ω has no compact connected components in O then the operator $T : \Sigma_1 \to \Sigma_2$ has a dense image.*

PROOF. We need to prove that restrictions to Ω of elements of $S(O) \cap W^{s,2}(E_{|O})$ are dense in $S(\Omega) \cap W^{s,2}(E_{|\Omega})$ in the norm of $W^{s,2}(E_{|\Omega})$. However, since the boundary of Ω is regular, $S(\overline{\Omega})$ is dense in $S(\Omega) \cap W^{s,2}(E_{|\Omega})$ in the norm of $W^{s,2}(E_{|\Omega})$ (see Tarkhanov [63], ch 4). On the other hand, the complement of Ω has no compact connected components in O, and hence the theorem of Runge implies that $S(\overline{O})$ is dense in $S(\overline{\Omega})$ (see the same book, theorem 11.26). Since $S(\overline{O}) \subset S(O) \cap W^{s,2}(E_{|O})$, and the natural topology in $S(\overline{O})$ is stronger than the induced topology from $W^{s,2}(E_{|O})$, we obtain the required result. □

From the proof of the lemma we can see how to understand the words "regular boundary". If $s \geq p$, the word "regular" means any boundary. And if $s < p$ then this means that the complement of Ω in every boundary point is sufficiently massive. The reader can get a more exact characterization from the book of Tarkhanov [63] (ch. 4).

LEMMA 6.3. *If the differential operator P satisfies the condition $(U)_S$ on X then the operator $T : \Sigma_1 \to \Sigma_2$ is injective.*

PROOF. Let $f \in \Sigma_1$ and $Tf = 0$. This means that the solution $f \in S(O)$ vanishes on the non-empty open subset Ω of O. Hence the property $(U)_S$ implies $f \equiv 0$ everywhere in O, which was to be proved. □

However the most important property of the operator T (in view of the application, via Theorem 5.2, of the theory of §1 to Problem 6.1) is the following.

LEMMA 6.4. *The operator $T : \Sigma_1 \to \Sigma_2$ is compact.*

PROOF. We need to show that the operator T maps any bounded set to a relatively compact set.

Let $K \subset \Sigma_1$ be a bounded set, that is, one can find a constant $C > 0$ such that $\|f\| < C$ for all $f \in K$. The image of K by the mapping T, that is, $T(K)$ is a relatively compact set if from any sequence $\{\mathcal{F}_j\} \subset T(K)$ one can extract a subsequence $\{\mathcal{F}_{jk}\}$ converging in Σ_2.

However if $\{\mathcal{F}_j\} \subset T(K)$ then $\mathcal{F}_j = f_{j|\Omega}$ where $\{f_j\} \subset K$. The sequence $\{f_j\}$ is bounded in the Hilbert space Σ_1. Therefore it contains a subsequence $\{f_{jk}\}$ which converges weakly to some element $f \in \Sigma_1$ (see Riesz and Sz.-Nagy [46], s.32). Certainly $\{f_j\}$ converges to f in the topology of the space $\mathcal{D}'(E_{|O})$.

We use now the Stiltjes-Vitaly theorem (see Hormander [16], 4.4.2) to conclude that $\{f_{jk}\}$ converges to f in the topology of the space $C^{\infty}_{loc}(E_{|O})$. We set $\mathcal{F} = f_{|\Omega}$, and $\mathcal{F}_{jk} = f_{jk|\Omega}$ then $\mathcal{F} \in \Sigma_2$ and $\{\mathcal{F}_{jk}\}$ converges to $\mathcal{F}$ in Σ_2, which was to be proved. □

We can formulate now the main result on existence of bases with double orthogonality.

THEOREM 6.5. *If $\Omega \Subset O$ is an open set with a regular boundary whose complement (in O) has no compact connected components in O then in the space $S(O) \cap W^{s,2}(E_{|O})$ there is an orthonormal basis $\{b_\nu\}_{\nu=1}^{\infty}$ whose restriction to Ω is an orthogonal basis in $S(\Omega) \cap W^{s,2}(E_{|\Omega})$.*

PROOF. We construct this basis by a method which will allow to obtain additional information about the corresponding eigen-value problem.

Let Π be the operator of orthogonal projection on Σ_1 in H_1. The à priori interior estimates for solutions of elliptic systems imply that the space Σ_1 (and Σ_2) is a Hilbert space with a reproducing kernel (see Aronszajn [4]). Hence Π is an integral operator with a kernel $\mathcal{K}(x, y) \in C^{\infty}_{loc}(E \boxtimes E_{|(O\times O)})$.

If $\{e_\nu\}_{\nu=1}^{\infty}$ is an orthonormal basis of the space $S(O) \cap W^{s,2}(E_{|O})$ then for all $x \in O$ we have $\mathcal{K}(x, .) = \sum_{\nu=1}^{\infty} e_\nu(x) \otimes e_\nu(.)$, where the series converges in the norm of $W^{s,2}(E \otimes E_{|O})$. As a series of (matrix-valued) functions of two variables $(x, y) \in O \times O$, this series converges uniformly on compact subsets of $O \times O$.

Thus, $\Pi\mathcal{F} = (\mathcal{F}, \mathcal{K}(x, .))_{H_1}$ $(\mathcal{F} \in H_1)$. Now simple calculations show that the operator $\Pi T^*T : H_1 \to H_2$ is integral. Namely,

$$(\Pi T^*T)\mathcal{F} = \int_\Omega \sum_{|\alpha|\le s} < *D^\alpha K(x, .), D^\alpha \mathcal{F} >_y dv \quad (\mathcal{F} \in H_1).$$

From Lemmata 6.2, 6.3 and 6.4, and the results of Example 1.9 the restriction of the operator ΠT^*T to Σ_1 is injective, compact, and self-adjoint operator in Σ_1.

Hence, if we denote by $\{b_\nu\}$ the countable complete orthonormal system of eigenvectors of the operator ΠT^*T on Σ_1 (corresponding to eigenvalues $\{\lambda_\nu\} \subset (0,1)$), $\{b_\nu\}$ is an orthonormal basis of the space Σ_1 and $\{Tb_\nu\}$ is an orthogonal basis in Σ_2.

Therefore $\{b_\nu\}$ is a system with the double orthogonality property, which was to be proved. □

For an element $\mathcal{F} \in \Sigma_1$ we shall denote by $c_\nu(\mathcal{F})$ $(\nu = 1, 2, ...)$ its Fourier coefficients with respect to the orthonormal system $\{b_\nu\}$ in Σ_1, that is, $c_\nu(\mathcal{F}) = (\mathcal{F}, b_\nu)_{H_1}$. And for an element $\mathcal{F} \in$ $Sigma_2$ we shall denote by $k_\nu(\mathcal{F})$ $(\nu = 1, 2, ...)$ its Fourier coefficients with respect to the orthogonal system $\{Tb_\nu\}$ in Σ_2, that is, $k_\nu(\mathcal{F}) = \frac{(\mathcal{F}, Tb_\nu)_{H_2}}{(Tb_\nu, Tb_\nu)_{H_2}}$. Then the principal property of bases with double orthogonality is the following.

Lemma 6.6. *For any element $\mathcal{F} \in \Sigma_1$ we have*

$$c_\nu(\mathcal{F}) = k_\nu(T\mathcal{F}) \quad (\nu = 1, 2, ...) \tag{6.1}$$

Proof. Using the calculations of Example 1.9 we obtain

$$c_\nu(\mathcal{F}) = (\mathcal{F}, \frac{1}{\lambda_\nu}(\Pi T^*T)b_\nu)_{H_1} = \frac{1}{\lambda_\nu}(T\mathcal{F}, Tb_\nu)_{H_2} = k_\nu(T\mathcal{F}),$$

which was to be proved. □

We formulate now the solvability condition for Problem 6.1. Let $\mathcal{G}\widetilde{f}$ be the Green integral (see (5.1) constructed from the "initial" data of the problem. As already we noted, the restriction of the section $\mathcal{G}\widetilde{f}$ to Ω belongs to the space Σ_2.

Lemma 6.7. *For $\nu = 1, 2, ...$*

$$k_\nu(\mathcal{G}\widetilde{f}) = \int_{\partial D} \sum_{j=0}^{p-1} < C_j k_\nu(\Phi(., y)), \widetilde{f}_j >_y ds. \tag{6.2}$$

Proof. This consists of direct calculations with the use of equality (5.1). □

In order to determine the coefficients $k_\nu(\mathcal{G}\widetilde{f})$ $(\nu = 1, 2, ...)$ it is not necessary to know the basis $\{Tb_\nu\}$ in Σ_2. It is sufficient only to know the coefficients of the decomposition of the fundamental matrix $(\Phi(., y)$ $(y \in \partial D)$ with respect to this series. The properties of the coefficients $k_\nu(\Phi(., y) \in C^\infty_{loc}(F^*_{|X\setminus\Omega})$ we shall discuss in §7.

THEOREM 6.8. *If the boundary of the domain D is sufficiently smooth then for the solvability of Problem 11.1 it is necessary and sufficient that $\sum_{\nu=1}^{\infty} |k_\nu(\mathcal{G}\widetilde{f})|^2 < \infty$;*

PROOF. Necessity. Suppose that Problem 6.1 is solvable. Then Theorem 5.2 implies that the solution $\mathcal{G}\widetilde{f}$ extends from O^+ to the whole domain O as a solution belonging $S(O) \cap W^{s,2}(E_{|O})$. Having denoted this extension $\mathcal{F}$ we obtain $\mathcal{F} \in \Sigma_1$ and $T\mathcal{F} = \mathcal{G}\widetilde{f}$ on Ω. Therefore taking into the consideration formula (6.1), and using Bessel's inequality we obtain

$$\sum_{\nu=1}^{\infty} |k_\nu(G\widetilde{f})|^2 = \sum_{\nu=1}^{\infty} |k_\nu(T\mathcal{F})|^2 = \sum_{\nu=1}^{\infty} |c_\nu(\mathcal{F})|^2 = \|F\|^2_{H_1} < \infty$$

which was to be proved.

Sufficiency. Conversely, let condition (6.3) hold. Then the theorem of Riesz and Fisher implies that there exists an element $\mathcal{F} \in \Sigma_1$ such that $c_\nu(\mathcal{F}) = k_\nu(\mathcal{G}\widetilde{f})$ for $\nu = 1, 2, ...$ Applying the operator T to the series $\mathcal{F} = \sum_{\nu=1}^{\infty} c_\nu(\mathcal{F}) b_\nu$ which converges in the norm of H_1, and taking into the consideration that the system $\{Tb_\nu\}$ is a basis in Σ_2, we have

$$T\mathcal{F} = \sum_{\nu=1}^{\infty} c_\nu(\mathcal{F}) T b_\nu =$$

$$= \sum_{\nu=1}^{\infty} k_\nu(\mathcal{G}\widetilde{f}) T b_\nu = \mathcal{G}\widetilde{f} \quad on \quad \Omega.$$

Hence $F \in S(O) \cap W^{s,2}(E_{|O})$, and the restrictions to Ω of the sections $\mathcal{F}$ and $\mathcal{G}\widetilde{f}$ coincide. Since the differential operator P satisfies the condition $(U)_S$ on X it follows that the solution $\mathcal{F}$ coincides with $\mathcal{G}\widetilde{f}$ everywhere in O. We conclude now (using Theorem 5.2) that Problem 6.1 is solvable, which was to be proved. □

In conclusion we consider 2 examples.

EXAMPLE 6.9. Aizenberg (see Aizenberg and Kytmanov [3]) studied the Cauchy problem for holomorphic functions of one variable, that is, in the case $P = d/d\overline{z}$, and $B_0 = 1$. He took as O the unit circle (with centre at zero) divided into 2 parts by a smooth hypersurface $S \subset B\backslash\{0\}$ and he denoted by D that part of this circle which did not contain zero. The system of holomorphic monomials z^ν $(\nu = 1, 2...)$ is an example of an orthogonal basis in the subspace of $L^2(O)$ which consists of the holomorphic functions. Moreover this holds for any circle with centre at 0. Thus, choosing as Ω some circle with centre at zero, contained in $O\backslash\overline{D}$, and normalizing the monomials $z^\nu (\nu = 1, 2...)$ in $L^2(O)$ we get a simple basis with double orthogonality. If a solution of the Cauchy problem is looked for in the class $L^2(D)$, and the "initial" datum is $f_0 \in L^2(S)$ then Green's integral could be constructed as $\frac{1}{2\pi\sqrt{-1}}\int_S \frac{f_0(\zeta}{\zeta-z}d\zeta$. Then Theorem 6.8 gives with small modifications the result of Aizenberg (see Aizenberg and Kytmanov [3]). We note that this theorem of Aizenberg (and also the remark following it) was a model example for us. □

Example 6.10. In the paper of Shlapunov [55] the Cauchy problem for harmonic functions of the class $L^2(D)$ was studied. The standard system $B_0 = 1$ and $B_1 = \partial/\partial\nu$ was taken as a Dirichlet system on dD. If O is a ball with centre at zero and, S is a smooth hypersurface in O, dividing this domain into 2 connected components $O^{\pm}$ so that zero belongs to O^+, the system $\{b_\nu\}$ with the double orthogonality property was constructed in an explicit form. This system corresponds to a special choice of Ω. Namely $\Omega \Subset O^+$ is a ball with centre at zero such that $\overline{\Omega} \Subset O^+$, and this basis consists of the homogeneous harmonic polynomials in $\mathbb{R}^n$. Also in this parer, it was supposed that the "initial data" $f_0, f_1 \in L^2(S)$. Then as $\widetilde{f}_j$ $(j = 0, 1)$ one can take their extensions by zero on $\partial D\backslash S$, and Green integral (5.1) is simply

$$\int_S (\Phi(x,.)f_1 - \partial/\partial\nu\Phi(x,.)f_0)ds.$$

Thus, Theorem 2.1 of Shlapunov [55] is a very special case of Theorem 6.8. □

æ

§7. The Carleman formula

In this section we consider the regularization aspect of Problem 5.1.

Problem 7.1. *It is required to find a solution* $f \in S(D) \cap W^{s,2}(E_{|D})$ *using known values* $B_j f \in W^{s-b_j-1/2,2}(G_{j|\overline{S}})$ $(0 \le j \le p-1)$ *on* S.

It is easy to see from Corollary 1.8 that side by side with the solvability conditions for Problem 5.1 $(q = 2)$ bases with double orthogonality give the possibility of obtaining a suitable formula (of Carleman) for the regularization of solutions. We shall illustrate this on example of Problem 7.1.

Let $\{b_\nu\}$ be the basis with double orthogonality, constructed in the previous section, in the space $(\Sigma_1 =)S(O) \cap W^{s,2}(E_{|O})$ such that the restriction of $\{b_\nu\}$ to Ω (that is, $\{Tb_\nu\}$) is an orthogonal basis of $(\Sigma_2 =)S(\Omega) \cap W^{s,2}(E_{|\Omega})$.

As above, we denote by $\{k_\nu(\Phi(.,y))\}$ the sequence of Fourier coefficients for the fundamental matrix $\Phi(.,y)$ $(y \in \Omega)$ with respect to the system $\{Tb_\nu\}$.

Lemma 7.2. *The sections* $k_\nu(\Phi(.,y))$ $(\nu = 1, 2...)$ *are continuous, together with their derivatives up to order* $(p-s-1)$, *on the whole set* X.

Proof. Though the restrictions to Ω of the columns of the fundamental matrix $\Phi(.,y)$ (for $y \in \Omega$) do not belong to the space Σ_2, for all $y \in X$ they do belong to

$W^{p-s-1,q}(E_{|\Omega})$ where $q < \frac{n}{n-1}$. Hence the scalar products

$$k_\nu(\Phi(.,y)) = \frac{(\Phi(.,y), Tb_\nu)_{\Sigma_2}}{(Tb_\nu, Tb_\nu)_{\Sigma_2}} =$$

$$(7.1) \qquad = \frac{1}{\lambda_\nu} \sum_{|\alpha|\le s} \int_\Omega < *D^\alpha b_\nu, D^\alpha b_\nu \Phi(.,y) >_y dv \quad (\nu = 1, 2...).$$

are defined for all $y \in X$. Since $b_\nu \in C^\infty_{loc}(E_{|O})$ we have $k_\nu(\Phi(.,y)) \in C^{p-s-1}_{loc}(F^*)$. And this was to be proved. □

Using formula (7.1) one can see that the sections $k_\nu(\Phi(.,y))$ $(\nu = 1, 2...)$ extend to the boundary of Ω from each side as infinitely differentiable sections (at least, if the boundary is smooth).

Lemma 7.3. *For any number $\nu = 1, 2, ...$ we have $P' k_\nu(\Phi(.,y)) = 0$ everywhere in $X \backslash \overline{\Omega}$.*

Proof. Since $P'\Phi' = 1$ on $\mathcal{E}'(E^*)$ then (7.1) implies that

$$P' k_\nu(\Phi(.,y)) = P'\Phi'(\chi_\Omega(*b_\nu)) = \chi_\Omega(*b_\nu) \quad (\nu = 1, 2, ...),$$

and this proves the statement. □

We introduce the following kernels $\mathfrak{C}^{(N)}$ defined for $(x,y) \in O \times X$ $(x \neq y)$:

$$(7.2) \qquad \mathfrak{C}^{(N)}(x,y) = \Phi(x,y) - \sum_{\nu=1}^{N} b_\nu(x) \otimes k_\nu(\Phi(.,y)) \quad (N = 1, 2, ...).$$

Lemma 7.4. *For any number $N = 1, 2, ...$ the kernels $\mathfrak{C}^{(N)} \in C_{loc}(E \boxtimes F)$ satisfy $P(x)\mathfrak{C}^{(N)}(x,y) = 0$ for $x \in O$, and $P'(y)\mathfrak{C}^{(N)}(x,y) = 0$ for $y \in X \backslash \Omega$ everywhere except on the diagonal $\{x = y\}$.*

Proof. Since $\{b_\nu\} \subset S(O)$, this immediately follows from Lemma 7.3. □

From the following lemma one can see that the sequence of kernels $\{\mathfrak{C}^{(N)}\}$ interpolated for real values $N \geq 0$ in a suitable way, for example in the piece-constant way, gives a special Carleman function for Problem 7.1 (see Tarkhanov [63], §25).

Lemma 7.5. *For any multi-index α, $D^\alpha_y \mathfrak{C}^{(N)}(.,y) \to 0$ in the norm of $W^{s,2}(E \otimes F^*_{y|O})$ uniformly with respect to y on compact subsets of $X \backslash \overline{O}$, and even $X \backslash O$ if $|\alpha| < p - s - n/2$.*

Proof. First, we notice that, if $y \in X \backslash \overline{O}$, every column of the matrix $\Phi(x.,y)$ is an element of the space Σ_1. Therefore using Lemma 6.6 we obtain $\mathfrak{C}^{(N)}(.,y) =$

$\Phi(.,y) - \sum_{\nu=1}^{N} c_\nu(\Phi(.,y))$. Differentiating this identity with respect to y we find the equality

$$D_y^\alpha \mathfrak{C}^{(N)}(.,y) = D_y^\alpha \Phi(.,y) - \sum_{\nu=1}^{N} b_\nu \otimes C_\nu(D_y^\alpha \Phi(.,y)) \quad (y \in X \backslash \overline{O}). \tag{7.3}$$

The correspondence $y \to D_y^\alpha \Phi(.,y)$ defines a continuous linear mapping of the topological space $X\backslash\overline{O}$ to the direct sum of k copies of the space Σ_1. Therefore for every column of the matrix $D_y^\alpha \Phi(.,y)$ its Fourier series with respect to the orthonormal basis $\{b_\nu\}$ converges in the norm of Σ_1 uniformly with respect to y on compact subsets of $X\backslash\overline{O}$ (see Shlapunov [55], Lemma 3.1). This proves the first part of the lemma. As for the second part, it is sufficient to use the same arguments because for $\alpha| < p-s-n/2$ the correspondence $y \to D_y^\alpha \Phi(.,y)$ defines a continuous linear mapping of the whole set $X\backslash\overline{O}$ to the direct sum of k copies of the space Σ_1.
□

We can formulate now the main result of the section. For $f \in S(D) \cap W^{s,2}(E_{|D}))$ we denote by $\widetilde{f} \in W^{s-b_j-1/2,2}(G_{j|\partial D})$ $(0 \le j \le p-1)$ some (arbitrary) extensions of the sections $B_j f$ from S to the whole boundary

Theorem 7.6 (Carleman's formula). *For any solution $f \in S(D) \cap W^{s,2}(E_{|D})$ the following formula holds:*

$$f(x) = -\lim_{N\to\infty} \int_{\partial D} \sum_{j=0}^{p-1} < C_j \mathfrak{C}^{(N)}(x,.), \widetilde{f}_j >_y ds \quad (x \in D). \tag{7.4}$$

Proof. Let $\mathcal{G}(\widetilde{f})$ be the Green integral constructed by formula (5.1). Theorem 6.8 implies that $\sum_{\nu=1}^{\infty} |k_\nu(\mathcal{G}(\widetilde{f})| < \infty$. Hence, from the theorem of Riesz and Fisher, there exists an element $\mathcal{F} \in S(O) \cap W^{s,2}(E_{|O})$ such that $c_\nu(\mathcal{F}) = k_\nu(\mathcal{G}\widetilde{f})$. In proving Theorem 6.8 we saw that this solution $\mathcal{F}$ is an extension of $\mathcal{G}\widetilde{f}$ from the domain O^+ to the whole domain O as a solution in $S(O) \cap W^{s,2}(E_{|O})$. Then Theorem 5.2 implies that the section $f'(x) = -\mathcal{G}(\widetilde{f})(x) + \mathcal{F}(x)$ $(x \in D)$ belongs to $S(D) \cap W^{s,2}(E_{|D})$, and satisfies $B_j f' = f$ $(0 \le j \le p-1)$ on S. Using (uniqueness) Theorem 2.8 we see that $f = f'$ everywhere in D. Hence

$$f(x) = -(\mathcal{G}\widetilde{f})(x) + \mathcal{F}(x) = -(\mathcal{G}\widetilde{f})(x) - \sum_{\nu=1}^{\infty} k_\nu(\mathcal{G}\widetilde{f}) b_\nu(x)) =$$

$$= -(\mathcal{G}\widetilde{f})(x - \lim_{N\to\infty} \sum_{\nu=1}^{N} k_\nu(\mathcal{G}\widetilde{f}) b_\nu(x)). \tag{7.5}$$

Putting in (7.5) the expressions for the coefficients $k_\nu(\mathcal{G}\widetilde{f})$ $(\nu = 1, 2, ...)$ which are given in Lemma 6.7 we obtain

$$f(x) = -\int_{\partial D} \sum_{j=0}^{p-1} < C_j \Phi(x, .), \widetilde{f}_j >_y ds-$$

$$-\lim_{N\to\infty} \left(\sum_{\nu=1}^{N} \int_{\partial D} \sum_{j=0}^{p-1} < C_j k_\nu(\Phi(x, .)), \widetilde{f}_j >_y ds \right) b_\nu(x) =$$

$$-\lim_{N\to\infty} \int_{\partial D} \sum_{j=0}^{p-1} < C_j \left(\Phi(x, .) - \sum_{\nu=1}^{N} b_\nu(x) \otimes k_\nu(\Phi(x, .)) \right), \widetilde{f}_j >_y ds =$$

$$= -\lim_{N\to\infty} \int_{\partial D} \sum_{j=0}^{p-1} < C_j \mathfrak{C}^{(N)}(x, .), \widetilde{f}_j >_y ds,$$

which was to be proved. □

We emphasize that the integral on the right hand side of formula (7.4) depends only on values of the expressions $B_j f$ $(0 \le j \le p-1)$ on S. Thus, this formula is a quantitative expression of (uniqueness) Theorem 2.8. However this gives much more than the uniqueness theorem because there is sufficiently complete information about the Carleman function $\mathfrak{C}^{(N)}$.

For harmonic functions of several variables Carleman formula (7.4) is first met, apparently, in [55].

Remark 7.7. The series $\sum_{\nu=1}^{\infty} k_\nu(\mathcal{G}\widetilde{f}) b_\nu$ (defining the solution $\mathcal{F}$) converges in the norm of the space $W^{s,2}(E_{|O})$. The Stiltjes-Vitali theorem (see Hormander [16], 4.4.2) implies now that it converges together with all its derivatives on compact subsets of O. Then, from formula (7.5), one can see that the limit in (7.4) is reached in the topology of the space $C^\infty_{loc}(E_{|O})$.

æ

§8. Examples for systems of the simplest type

The examples of this section are based on the following simple observation.

LEMMA 8.1. *If the coefficients of the differential operator P are real analytic then Problem 5.1 is solvable if and only of the section $\mathcal{G}(\widetilde{f})$ extends from O^+ to the whole domain O as a real analytic section belonging to $W^{s,q}(E_{|D})$.*

PROOF. First, we note that, since $P\mathcal{G}(\widetilde{f}) = 0$ outside of ∂D, the section $\mathcal{G}(\widetilde{f})$ is real analytic in the domain O^+. Now let $\mathcal{F}$ be the above extension of this section in O. Then $P\mathcal{F}$ is also a real analytic section in O, and $P\mathcal{F} = 0$ in O^+. From the uniqueness theorem we obtain that $P\mathcal{F} = 0$ everywhere in the domain O, that is, $\mathcal{F} \in S(O) \cap W^{s,2}(E_{|O})$. Therefore the statement of the lemma follows from Theorem 5.2. □

In particular, we can use the fact that $(P^*P)\mathcal{G}\widetilde{f} = 0$ everywhere outside ∂D, and the extendibility condition for $\mathcal{G}(\widetilde{f})$ (up to a section $\mathcal{F} \in W^{s,2}(E_{|O})$ satisfying $(P^*P)\mathcal{F} = 0$ in O) write in the language of bases with the double orthogonality.

DEFINITION 8.2. The differential operator P is said to be a simplest type operator if $p = 1$, and $P^*P = -\Delta I_k$ where Δ is the Laplace operator in $\mathbb{R}^n$.

We suppose that P is a (elliptic) differential operator of the simplest type in $\mathbb{R}^n$ (see §8). Let $O = B_R$ be the ball in $\mathbb{R}^n$ with centre at zero and of radius $0 < R < \infty$, and S be a smooth closed hypersurface in B_R dividing this ball into 2 connected components O^+, and $D = O^-$ so that the domain O^+ contains zero. We consider the following problem (of Cauchy).

PROBLEM 8.3. *Let $f_0 \in C_{loc}(E_{|S})$ be a summable section of E on S. It is required to find a solution $f \in S(D) \cap C_{loc}(E_{|D\cup S})$ such that $f_{|S} = f_0$.*

As the fundamental solution of the differential operator P we can take the matrix $\Phi(x, y) = P'(y)g(x - y)$, where $g(x - y)$ is the standard fundamental solution of convolution type of the Laplace operator in $\mathbb{R}^n$ with the opposite sign. Then the Green integral (5.1) has the following form:

$$\mathcal{G}\widetilde{f}(x) = \frac{1}{\sqrt{-1}} \int_S \Phi(x, .)\sigma(P)(\nu) f_0 ds \ (x \notin S).$$

It is easy to see from the structure of the fundamental matrix Φ that the components of the section $\mathcal{G}\widetilde{f}$ are harmonic functions everywhere in B_R (and even in $\mathbb{R}^n$) except on the set S.

We need a basis with the double orthogonality in the subspace of $L^2(B_R)$ which consists of harmonic functions. In [51] this closed subspace of $L^2(B_R)$ with the induced hermitian structure was denoted by $h^2(B_R)$. Let $\{h^{(i)}_\nu\}$ be a set of homogeneous harmonic polynomials which form a complete orthonormal system in $L^2(\partial B_R)$ where ν is the degree of homogeneity, and i is an index labelling the polynomials of degree ν belonging to the basis. The size of the index set for i as a function of ν is known, namely, $1 \le i \le J(\nu)$ where $J(\nu) = \frac{(n+2\nu-2)(n+\nu-3)!}{\nu!(n-2)!}$.

LEMMA 8.4. *For any $0 < r < \infty$ the system $\{\sqrt{\frac{n+2\nu}{r^{n+2\nu}}} h^{(i)}_\nu\}$ is an orthonormal basis in $h^2(B_r)$ and an orthogonal basis in $h^2(B)$ where B is an arbitrary ball with centre at zero.*

PROOF. See Shlapunov [55], Lemma 3.5. □

We fix $0 < r < dist(0, S)$ and set $\Omega = B_r$ so that $\Omega \Subset O$. It easy to see from Lemma 8.4 that for any $0 < R < \infty$ the system $\{\sqrt{\frac{n+2\nu}{R^{n+2\nu}}} h^{(i)}_\nu\}$ is an orthonormal basis in $h^2(B_R)$ and an orthogonal basis in $h^2(B_r)$. In order to obtain the Fourier coefficients for the section $\mathcal{G}(\widetilde{f})$ with respect to this basis in $h^2(B_r)$ it is sufficient to know the Fourier coefficients for the fundamental matrix $\Phi(x,y)$ (see (6.2)). The information about them is contained in the following lemma.

LEMMA 8.5.

$$\Phi(x,y) = \Phi(0,y) - \sum_{\nu=1}^{\infty}\sum_{i=1}^{J(\nu)} h_\nu^{(i)}(x){P^*}'(y)[\frac{1}{n+2\nu-2}\frac{\overline{h_\nu^{(i)}(y)}}{|y|^{n+2\nu-2}}]. \tag{8.2}$$

where the series converges together with all the derivatives uniformly on compact subsets of the cone $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \,:\, |y| > |x|\}$.

PROOF. It is sufficient to use the similar decomposition for $g(x-y)$ which was found for even $n > 2$ by Kytmanov (see Aizenberg and Kytmanov [3]) and for the general case by Shlapunov [55] (Lemma 3.2), and then to use the equality $\Phi(x,y) = P'(y)g(x-y)$. $\square$

Our principal result will be formulated in the language of the coefficients

$$k_\nu^{(i)} = \frac{1}{\sqrt{-1}}\int_S {P^*}'(y)[\frac{1}{n+2\nu-2}\frac{\overline{h_\nu^{(i)}(y)}}{|y|^{n+2\nu-2}}]\sigma(P)(\nu)f_0 ds \ (\nu = 1,2,...).$$

THEOREM 8.6. *For solvability of Problem 8.3, it is necessary and sufficient that*

$$\limsup_{\nu\to\infty} \max_i \sqrt[\nu]{|k_\nu^{(i)}(y)|} \leq \frac{1}{R} \tag{8.4}$$

PROOF. Necessity. Let Problem 8.3 be solvable. Then Theorem 5.2 implies that the solution $\mathcal{G}\widetilde{f}^+$ on the domain O^+ extends to a solution $\mathcal{F}$ on the whole ball B_R.

We fix $0 < r < R$. It is clear that the components of the solution $\mathcal{F}$ belong to the space $h^2(B_r)$. Therefore, from Lemma 8.4, they are represented by their Fourier series with respect to the system $\{\sqrt{\frac{n+2\nu}{r^{n+2\nu}}}h_\nu^{(i)}\}$

$$\mathcal{F}(x) = \sum_{i,\nu} c_\nu^{(i)}(r)\sqrt{\frac{n+2\nu}{r^{n+2\nu}}}h_\nu^{(i)}(x) \quad (x \in B_r). \tag{8.5}$$

Bessel's inequality implies that the series $\sum_{i,\nu}|c_\nu^{(i)}(r)|^2$ converges. On the other hand, in the ball Ω,from Lemma 8.5, we obtain the decomposition

$$\mathcal{G}\widetilde{f}(x) = \sum_{i,\nu} k_\nu^{(i)}h_\nu^{(i)}(x) \quad (x \in \Omega). \tag{8.6}$$

Comparing (8.5) and (8.6) we find that

$$c_\nu^{(i}(r) = \sqrt{\frac{r^{n+2\nu}}{n+2\nu}} k_\nu^{(i)} \quad (\nu = 1, 2, ...). \tag{8.7}$$

Hence for any $0 < r < R$

$$\sum_{i,\nu} |k_\nu^{(i)}(r)|^2 \frac{r^{n+2\nu}}{n+2\nu} = r^n \sum_{\nu=0}^{\infty} \left(\sum_{i=1}^{J(\nu)} \frac{|k_\nu^{(i)}(r)|^2}{n+2\nu} \right) r^{2\nu} < \infty$$

Using the Cauchy-Hadamard formula for the radius of the convergence of a power series we obtain

$$\limsup_{\nu\to\infty} \max_i \sqrt[\nu]{|k_\nu^{(i)}(y)|} \leq \limsup_{\nu\to\infty} \left(\sum_{i=1}^{J(\nu)} \frac{|k_\nu^{(i)}(r)|^2}{n+2\nu} \right)^{1/2\nu} \leq \frac{1}{r}$$

Since $0 < r < R$ is arbitrary then condition (8.4) holds, which was to be proved.

Sufficiency. If condition (8.4) holds then the Cauchy-Hadamard formula and the estimate $J(\nu) < const\, \nu^{n-2}$ implies that the series $\sum_{i,\nu} |k_\nu^{(i)}(r)|^2 \frac{r^{n+2\nu}}{n+2\nu}$ converges for any $0 < r < R$. The Riesz-Fisher theorem implies that there exists a section $\mathcal{F}$ (of the bundle $E_{|B_r}$) with the components from $h^2(B_r)$ such that

$$\mathcal{F}(x) = \sum_{i,\nu} \sqrt{\frac{r^{n+2\nu}}{n+2\nu}} k_\nu^{(i)} \sqrt{\frac{n+2\nu}{r^{n+2\nu}}} h_\nu^{(i)}(x) =$$

$$= \sum_{i,\nu} k_\nu^{(i)} h_\nu^{(i)}(x)$$

where the series converges in the norm of the space $L^2(E_{B_r})$. It is easy to see that in the ball Ω the section $\mathcal{F}$ coincides with $\mathcal{G}\widetilde{f}$. Therefore it is a harmonic (and hence real analytic) extension of the Green integral $\mathcal{G}\widetilde{f}$ from O^+ to the whole domain O. Now using Lemma 8.1 and Corollary 2.5 we can conclude that Problem 8.3 is solvable. This proves the theorem. □

In conclusion we give the corresponding variant of Carleman's formula. For each number $N = 1, 2...$ we consider the kernel $\mathfrak{C}^{(N)}(x, y)$ defined, for all $y \neq 0$ off the diagonal $\{x = y\}$, by the equality

$$\mathfrak{C}^{(N)}(x, y) = \Phi(x, y) - \Phi(0, y) + \sum_{\nu=1}^{N} \sum_{i=1}^{J(\nu)} h_\nu^{(i)}(x) P^{*'}(y) [\frac{1}{n+2\nu-2} \frac{\overline{h_\nu^{(i)}(y)}}{|y|^{n+2\nu-2}}]. \tag{8.8}$$

LEMMA 8.7. *For any number $N = 1, 2, ...,$ the kernel $\mathfrak{C}^{(N)}$ is an infinitely differentiable section of $E \boxtimes F$, which is harmonic with respect to x, and satisfying $P'(y)\mathfrak{C}^{(N)}(x,y) = 0$ for all $y \neq 0$ off the diagonal $\{x = y\}$.*

PROOF. This follows from the properties of the matrix Φ and the polynomials $h_\nu^{(i)}(y)$. □

We note that since $\mathfrak{C}^{(N)}$ is a "remainder" summand in the formula (8.2), $\mathfrak{C}^{N)}(x,y) \to 0$ $(N \to \infty)$, together with all its derivatives uniformly on compact subsets of the cone $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : |y| > |x|\}$.

THEOREM 8.8 (CARLEMAN'S FORMULA). *For any solution $f \in S(D) \cap C_{loc}(E_{|D\cup S})$ whose restriction to S is summable there the following formula holds*

$$f(x) = -\lim_{N\to\infty} \int_S \mathfrak{C}^{(N)}(x,.)\sigma(P)(\nu) f_0 ds \quad (x \in D). \tag{8.9}$$

PROOF. This is similar to the proof of Theorem 7.6. □

For the specific domain D bounded by a part of the surface of a cone and a piece of a smooth hypersurface S which is contained in the cone explicit Carleman formulae in form (8.9) were obtained earlier in the papers of Jarmuhamedov [18], and his students (see Mahmudov [36], and others).

Remark 8.9. As in Theorem 7.6, the convergence of the limit in (8.9) is uniform on compact subsets of the domain D together with all its derivatives.

æ

æ

PART II.

THE GENERAL CASE

INTRODUCTION

We continue to consider the Cauchy problem for solutions of the system $Pf = 0$ where $P \in do_p(E \to F)$ is some differential operator with an injective symbol on an open set $X \subset \mathbb{R}^n$ (see part 1), and $E = X \times \mathbb{C}^k$, $F = X \times \mathbb{C}^l$ are (trivial) vector bundles over X whose sections of the class $\mathfrak{C}$ over an open set $\sigma \subset X$ are interpreted as columns of functions from $\mathfrak{C}(\sigma)$, that is, $C(E_{|\sigma}) = [\mathfrak{C}(\sigma)]^k$ and similarly for F.

We shall often use notation from part 1 of this paper without special explanations.

We suppose that the differential operator P has real analytic coefficients. It is known that in this case there is for the differential operator P a complex of compatibility conditions, $\{E^i, P^i\}$ say, in which the differential operators $P^i \in do_{p_i}(E^i \to E^{i+1})$ also have real analytic coefficients (see Dudnikov and Samborskii [10], §9).

Let $D \Subset X$ be a domain with a boundary of class C^p_{loc} (for $p = 1$ we require that $\partial D \in C^2_{loc}$). For some of the results of this paper higher smoothness of the boundary is required, but it is always sufficient that $\partial D \in C^\infty_{loc}$.

We fix a Dirichlet system of order $(p-1)$ on ∂D, say, $B_j \in do_{b_j}(E \to G_j)$ $(0 \le j \le p-1)$ where $G_j = U \times C^k$ are (trivial) vector bundles over a sufficiently small neighbourhood U of the boundary of the domain D.

Problem 1. *Let f_j $(0 \le j \le p-1)$ be given sections of the bundles G_j over an (open) set $S \subset \partial D$. It is required to find a solution $f \in S^f(D)$ such that $B_j f_{|S} = f_j$ $(0 \le j \le p-1)$.*

Unlike part 1, here we concentrate on the situation where P is an overdetermined operator, i.e. $l > k$, though the case $l = k$ is also formally permitted. What new facts does this bring to Problem 1 ?

First, the differential operator P may have no right fundamental solution. Hence the Green integral $\mathcal{G}\widetilde{f}$ (see part 1, (5.1)) may, perhaps, not satisfy the equation $P\mathcal{G}\widetilde{f} = 0$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

On the other hand, every overdetermined differential operator P induces on the hypersurface S a tangential differential operator P_b, and now "the initial data" $(\oplus f_j)$ must satisfy the induced tangential equation on S (see Tarkhanov [64], §11). We denote by $\{C_j\}_{j=0}^{p-1}$ the Dirichlet system of order $(p-1)$ on ∂D associated to the system $\{B_j\}$ in the Green formula for the differential operator P. This system is determined in a natural way in Lemma 2.3 (see part 1).

LEMMA 2. *If Problem 1 is solvable then $P_b(\oplus f_j) = 0$ (weakly) on S, that is,*

$$(1) \quad \int_S < C_j(P'v), f_j >_y ds = 0 \quad \text{for all } v \in \mathcal{D}(E^{2'}) \text{ such that } (\operatorname{supp} v) \cap \partial D \subset S.$$

PROOF. Let there be a solution $f \in S^f(D)$ such that $B_j f = f_j$ $(0 \le j \le p-1)$ on S. Then, if $v \in \mathcal{D}(E^{2'})$ and $(\operatorname{supp} v) \cap \partial D \subset S$, the Stokes formula implies

$$\int_S < C_j((P^1)'v), f_j >_y ds = \int_{\partial D} < C_j((P^1)'v), B_j f >_y ds =$$

$$= \lim_{\varepsilon \to +0} \int_{\partial D_\varepsilon} G_P((P^1)'v), f) = 0,$$

which was to be proved. □

In §9 we show how Problem 1 may be reduced to the Cauchy problem for solutions of elliptic systems which was considered in part 1 of this paper.

In §10 we prove a solvability criterion for the Cauchy problem for systems with injective symbol in terms of the Green integral. By using "Cauchy data" on S we construct the Green integral which satisfies $P^*Pf = 0$ everywhere outside of an arbitrary small neighbourhood of S on ∂D. Then the Cauchy problem is solvable if and only if this integral analytically extends across S from the complement of D to this domain with preservation of a suitable Sobolev class, and the Cauchy data on S satisfy the tangential equation on S.

In §11 the condition for extendibility (as a solution of the system $P^*Pf = 0$) across S of Green s integral is written in terms of space bases with double orthogonality. As in §6, their construction depends on solution of an eigenvalue problem for a compact self- adjoint operator. So this fragment of the application of bases with double orthogonality is most similar to the original Bergman concept [6] (see part 1).

The use of bases with double orthogonality not only gives information about solvability conditions for the Cauchy problem. It also leads to visible formulae for regularization. A Carleman function of the Cauchy problem for solutions of systems with injective symbols is constructed in §12.

Finally, in §13 we consider some examples of differential equations of the simplest type including the many dimensional Cauchy-Riemann system. More exactly we extend the results of §8 about elliptic systems of the simplest type to overdetermined systems of the simplest type. In particular, this section includes the results of Aizenberg and Kytmanov [3].

§9. Reduction of the Cauchy problem for systems with injective symbols to the Cauchy problem for elliptic systems

Let $O \Subset X$ be a domain and S be a smooth closed hypersurface in O dividing this domain into two connected components: $O^- = D$ and $O^+ = O \backslash \overline{D}$. For our purposes, it is sufficient to consider that the Dirichlet system $\{B_j\}$ is given only in some neighbourhood of (compact) S.

We recall the definition of the operator $*$ which acts on the bundles E,F and G_j $(0 \leq j \leq p-1)$. We endow each of these bundles, which is abstractly denoted by B, with some hermitian metric $(.,.)_x$. Then $* : B \to B^*$ is a conjugate linear isomorphism of bundles given by means of $< *\varphi, f >_x = (f, \varphi)_x$ $(f \in B_x)$.

Also P' is the transposed operator, and $P^* = *^{-1} P' *$ is the formally adjoint operator for the differential operator P.

Lemma 9.1. *The differential operator $\Delta = P^* P$ has a (bilateral) fundamental solution $\mathcal{J} \in pdo_{-2p}(E \to E)$ whose kernel is real analytic off the diagonal of $X \times X$.*

Proof. This follows from the theorem of Malgrange (see Tarkhanov [64], §8) because Δ is an elliptic differential operator of order $2p$ with real analytic coefficients on X. □

We consider the following system of boundary operators defined in the neighbourhood U of the boundary ∂D. For a section $f \in C^{p-1}_{loc}(E_{|U})$ we set $\tau(f) = \oplus(B_j f)$, that is, $\tau(f)$ is a representation of the Cauchy data on S with respect to the differential operator P. Similarly for $g \in C^{p-1}_{loc}(F_{|U})$ we set $\nu(g) = \oplus(*^{-1} C_j * g)$, that is, $\nu(g)$ represents the Cauchy data of g on S with respect to the differential operator P^*.

Lemma 9.2. *The system of boundary operators $\{\tau(.), \nu(P.)\}$ forms a Dirichlet system of order $(2p-1)$ on ∂D.*

Proof. This fact has already been noted in the proof of Theorem 4.4. (see part 1), and it is proved by simple calculations. □

For easy reference we note a simple consequence of Theorem 2.6.

Lemma 9.3. *Let $S \in C^\infty_{loc}$. Then, for any solution $f \in S^f(O^\pm)$ which has finite order of growth near S, the expressions $\tau(f)$ and $\nu(Pf)$ have weak limit values on S belonging to $\mathcal{D}'(\oplus G_{j|S})$.*

Proof. The statement of the lemma follows from Theorem 2.6 and Lemma 9.2 because, for any domain $D' \subset O^\pm$ whose boundary intersects the boundary of $O^\pm$ only in the set S, the restriction of the solution f on D' belongs to $S^f_\Delta(D')$, and because it is possible to extend the Dirichlet system $\{\tau(.), \nu(P.)\}$ from $\partial D' \cap S$ to the whole boundary $\partial D'$ as a suitable Dirichlet system (at least, if the boundary of $\partial D'$ is sufficiently smooth). □

We could not prove the converse statement (as we did in Theorem 2.6) except in the case when S is a connected component of the boundary of the domain $O^{\pm}$.

LEMMA 9.4. *Let $S \in C^{\infty}_{loc}$. If the solutions $f^{\pm} \in S_{\Delta}(O^{\pm})$ have finite orders of growth near S, and $\tau(f^+) = \tau(f^-)$ and $\nu(Pf^+) = \nu(Pf^-)$ on S then there is a solution $f \in S_{\Delta}(O)$ such that $f_{|O^{\pm}}) = f^{\pm}$.*

PROOF. It is sufficient to use Theorem 3.2 from the book of Tarkhanov [62] taking into consideration Lemma 9.2. □

The following theorem for the Cauchy - Riemann system in the space $\mathbb{C}^n$ was first proved, apparently, by Kytmanov (see Aizenberg and Kytmanov [3]).

THEOREM 9.5. *We suppose that $S \in C^{\infty}_{loc}$. If a solution $f \in S_{\Delta}(D)$ has finite order of growth near S, and $P_b(\tau(f)) = 0$, and $\nu(Pf) = 0$ on S then $Pf = 0$ everywhere in the domain D.*

PROOF. Let the solution $f \in S_{\Delta}(D)$ have finite order of growth near the hypersurface S. Then, from Lemma 9.3, the expressions $\tau(f)$ and $\nu(Pf)$ have weak limit values on S belonging to $\mathcal{D}'(\oplus G_{j|S})$. We suppose that $P_b(\tau(f)) = 0$, and $\nu(Pf) = 0$ on S.

Fix an arbitrary point $x^0 \in S$. Since the differential operator P has an injective symbol then the complex of compatibility conditions $\{E^i, P^i\}$ (which is induced by P) is exact in positive degrees on the level of sheaves over X. In particular, this means that for any neighbourhood $U = U(x^0)$ of the point x^0 and any section $f \in S_{P^1}(U)$ there exist a possibly smaller neighbourhood $V = V(x^0)$ of this point, and a section $u \in C^{\infty}_{loc}(E_{|V})$ such that $Pu = f$ on V (see Tarkhanov [64], Theorem 3.10).

Since $\tau(f)$ represents the Cauchy data of f on S with respect to the differential operator P, and $P_b(\tau(f)) = 0$ on S then the exact Mayer -Vietoris sequence (see Theorem 18.9 in the book of Tarkhanov [64]) implies that there are a neighbourhood $V = V(x^0)$ of the point x^0 in O and solutions $f^{\pm} \in S_{\Delta}(O^{\pm} \cap V)$ having finite order of growth near $S \cap V$ such that $\tau(f^+) - \tau(f^-) = \tau(f)$ on $S \cap V$.

Consider now two sections $\mathcal{F}^+ = f^+$ and $\mathcal{F}^- = f^- + f$ defined on the open sets $O^+ \cap V$ and $O^- \cap V$ respectively.

By construction, the sections $\mathcal{F}^{\pm} \in S_{\Delta}(O^{\pm} \cap V)$ have finite orders of growth near the hypersurface $S \cap V$, and $\tau(\mathcal{F}^+) = \tau(\mathcal{F}^-)$, and $\nu(P\mathcal{F}^+) = 0 = \nu(P\mathcal{F}^-)$ on $S \cap V$. Hence we can use Lemma 9.4, and conclude that there exists a section $\mathcal{F} \in S_{\Delta}(V)$ such that $\mathcal{F}_{|O^{\pm} \cap V} = \mathcal{F}^{\pm}$.

The differential operator Δ is elliptic and has real analytic coefficients therefore the theorem of Petrovskii implies that the sections $\mathcal{F}$ and $P\mathcal{F}$ are real analytic in V. Since $P\mathcal{F} = 0$ in $O^+ \cap V$, we can conclude that $P\mathcal{F} = 0$ everywhere in V.

Thus, $Pf = P\mathcal{F} - P\mathcal{F}^- = 0$ in $D \cap V$, and f is real analytic in the domain D. Hence we have $Pf = 0$ everywhere in this domain which was to be proved. □

We note that without the requirement "$P_b(\tau(f)) = 0$ on S" Theorem 9.5 is false.

EXAMPLE 9.6. Let $P(D) = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \cdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}$ be the gradient operator in $\mathbb{R}^n$ ($n > 1$), and $B_0 = 1$. Then $\Delta = P^*P$ is (minus) the usual Laplace operator in $\mathbb{R}^n$, and $\tau(f) = f$, and $\nu(Pf) = \frac{\partial f}{\partial \nu}$. In particular, if S is a piece of the hypersurface $\{x_n = 0\}$, any harmonic function f in D which does not depend on the variable x_n satisfies $\nu(Pf) = 0$ on S. But, certainly, such a function may be non-constant in D. □

At the same time, if $S = \partial D$ then the condition "$P_b(\tau(f)) = 0$ on an open subset of S" in Theorem 10.3 is not necessary (see Karepov and Tarkhanov [20]).

Remark 9.7. As one can see from the proof of Theorem 2.6, the smoothness condition for the hypersurface S in Lemmata 9.3, 9.4, and Theorem 9.5 can be loosened if we consider à priori solutions of the system $Pf = 0$ of order of growth which is not greater than a given fixed number. But this is a general observation. □

Theorem 9.5 gives a method of studying Problem 1. More precisely it shows that this problem is equivalent to the Cauchy problem for solutions of the system $P^*Pf = 0$ with initial data $\tau(f) = \oplus f_j$ and $\nu(Pf) = 0$ on S. The last problem belongs already to the range of Cauchy problems for elliptic systems which was considered in part 1 of this paper.

In the following sections we realize this method. æ

§10. The Green integral and solvability of the Cauchy problem for systems with injective symbols

We formulate Problem 1 more precisely (as we did in §5).

PROBLEM 10.1. *Let* $f_j B^{s-b_j-1/q,q}(G_{j|\overline{S}})$ $(0 \le j \le p-1)$ *be known sections on* S *where* $s \in \mathbb{Z}_+$, *and* $1 < q < \infty$. *It is required to find a section* $f \in S(D) \cap W^{s,q}(E_{|D})$ *such that* $B_j f = f_j$ $(0 \le j \le p-1)$ *on* S.

Using the "initial" data of Problem 10.1 we construct the Green integral in a special way.

Namely, as a left fundamental solution of the differential operator P we take the kernel $\Phi(x,y) = P^{*'}\mathcal{J}(x,y)$ where $\mathcal{J}$ is a fundamental solution of the "laplacian" $\Delta = P^*P$ about which we spoke in Lemma 9.1.

We denote by $\widetilde{f} \in B^{s-b_j-1/q,q}(G_{j|\partial D})$ $(0 \le j \le p-1)$ an extension of the section f_j to the whole boundary. If, for example, $s = 0$ and $f_j \in L^2(G_{j|S})$ $(0 \le j \le p-1)$, it is possible to extend them by zero on $\partial D \backslash S$. In any case the extensions could be chosen so that they will be supported on a given neighbourhood of the compact S on ∂D. Then we set $\widetilde{f} = \oplus f_j$, and

$$\mathcal{G}(\widetilde{f})(x) = \int_{\partial D} < C_j \Phi(x,.), \widetilde{f}_j >_y ds \quad (x \in \partial D) \tag{10.1}$$

Lemma 10.2. *The potential* $\mathcal{G}(\widetilde{f})$ *satisfies* $\Delta\mathcal{G}(\widetilde{f}) = 0$ *on each of the open sets* D *and* $X\backslash\partial D$*, and has finite order of growth near the surface* ∂D*.*

Proof. This follows from equality (10.1) and the structure of the fundamental solution $\Phi(x,y)$. □

In particular, if we denote by $\mathcal{F}^{\pm}$ the restrictions of the section $F \in D'(E_{|O})$ to the sets $O^{\pm}$, we have $\mathcal{G}(\widetilde{f})^{\pm} \in S_{\Delta}(O^{\pm})$.

Theorem 10.3. *If the boundary of the domain* D *is sufficiently smooth then, for Problem 10.1 to be solvable, it is necessary and sufficient that*

(1) *the integral* $\mathcal{G}(\widetilde{f})$ *extends from* O^{+} *to the whole domain* O *as a solution belonging to* $S_{\Delta}(O) \cap W^{s,q}(E_{|O})$*;*
(2) $P_b(sf) = 0$ *in a neighbourhood of some point* x^0 *on* S*.*

Proof. Necessity. Suppose that there is a section $f \in S(D) \cap W^{s,q}(E_{|D})$ such that $B_j f = f_j$ $(0 \le j \le p-1)$ on S.

We consider in the domain O (more exactly, in $O\backslash S$) the following section:

$$\mathcal{F}(x) = \begin{cases} \mathcal{G}\widetilde{f}(x), x \in O^{+}, \\ \mathcal{G}\widetilde{f}(x) + f(x), x \in O^{-}. \end{cases} \tag{10.2}$$

Using the boundedness theorem for potential operators in Sobolev spaces on manifolds with boundary (see Rempel and Schulze [45], 2.3.2.5) we can conclude that $\mathcal{G}(\widetilde{f})^{\pm} \in W^{s,q}(E_{|O^{\pm}})$ (if the surface ∂D is sufficiently smooth, for example if $\partial D \in C^r$, $r = max(s, p-s)$). This means that $\mathcal{F}^{\pm} \in W^{s,q}(E_{|O^{\pm}})$.

On the other hand, we consider the difference $\delta = \mathcal{G}(\oplus B_j f) - \mathcal{G}(\widetilde{f})$. Let $\varphi_\varepsilon \in D(X)$ be any function supported on the ε-neighbourhood of the set $\partial D\backslash S$, and being equal to 1 in some smaller neighbourhood of this set. Since $B_j f = \widetilde{f}_j$ $(0 \le j \le p-1)$ on S then we can write

$$\delta(x) = \int_{\partial D} \sum_{j=0}^{p-1} < C_j\Phi(x,.), \varphi_\varepsilon(B_j f - \widetilde{f}_j) >_y ds (x \notin \partial D).$$

The right hand side of this equality is a solution of the system $\Delta f = 0$ everywhere in the domain O except the part of the ε-neighbourhood of the boundary of S on ∂D which belongs to O. Therefore, since $\varepsilon > 0$ is arbitrary, $\delta \in S_{\Delta}(O)$.

Now expressing the integral $\mathcal{G}(\oplus B_j f)$ from the Green formula (2.3) (see part 1) and putting $\mathcal{G}(\widetilde{f}) = \mathcal{G}(\oplus B_j \widetilde{f}) - \delta$ in inequality (10.2) we obtain

$$\mathcal{F}(x) = -\delta(x) \quad (x \in O\backslash S)$$

Hence the section $\mathcal{F}$ extends to the whole domain O as a solution of the system $\Delta f = 0$.

Thus $\mathcal{F}$ belongs to $S_\Delta(O)\cap W^{s,q}(E_{|O})$, and on O^+ this section coincides with $\mathcal{G}(\widetilde{f})^+$, which was to be proved.

Sufficiency. Conversely, let $\mathcal{F}\in S_\Delta(O)\cap W^{s,q}(E_{|O})$ be a solution coinciding with $\mathcal{G}(\widetilde{f})^+$ on O^+, and $P_b(\oplus f_j)=0$ in a neighbourhood of some point x^0 on S.

We set $f(x) = -\mathcal{G}(\widetilde{f}) + \mathcal{F}(x)$ $(x\in D)$. The above mentioned boundedness theorem for potential operators in Sobolev spaces (see Rempel and Schulze [45], 2.3.2.5) implies that $\mathcal{G}(\widetilde{f})\in W^{s,q}(E_{|O^-})$. Therefore $f\in S_\Delta(D)\cap W^{s,q}(E_{|D})$, and f has finite order of growth near the hypersurface S.

Now Lemma 2.7 (see part 1) on the weak jump of the Green integral associated with the differential operator Δ and the Dirichlet system $\{\tau(.),\nu(P.)\}$ on ∂D implies that

$$\begin{cases} \tau(\mathcal{G}\widetilde{f}(x)^+) - \tau(\mathcal{G}\widetilde{f}(x)^-) = \oplus\widetilde{f}_j \text{ on } \partial D,\\ \nu(P\mathcal{G}(\widetilde{f})^+) - \nu(P\mathcal{G}(\widetilde{f})^-) = 0 \text{ on } \partial D.\end{cases}$$

Since $\tau(\mathcal{G}(\widetilde{f})^+) = \tau(\mathcal{F})$, and $\nu(P\mathcal{G}(\widetilde{f})^+) = \nu(P\mathcal{F})$ on S then these equations imply that

$$\begin{cases} \tau(f) = \oplus\widetilde{f}_j \text{ on } S,\\ \nu(Pf) = 0 \text{ on } S.\end{cases}$$

We use now the condition "$P_b(\oplus f_j)=0$ in a neighbourhood $V=V(x^0)$ on S". Then $P_bf(\tau(f))=0$ in V, and, from Theorem 9.5 applied to the piece $V\cap S$ instead of S, we obtain that $Pf=0$ everywhere in the domain D.

Hence $f\in S(O)\cap W^{s,q}(E_{|O})$ is the required solution of Problem 10.1, which was to be proved. □

For the Cauchy-Riemann operator in $\mathbb{C}^n$ $(n>1)$ Theorem 10.3 is due to Aizenberg and Kytmanov (see [3], and also Aizenberg [2]).

There is an example showing that the sufficiency part of Theorem 10.3 without the requirement "$P_b(\oplus\widetilde{f}_j)=0$ on an open subset of S" is false.

Example 10.4. Let $P(D) = \begin{pmatrix}\frac{\partial}{\partial x_1}\\ \cdots\\ \frac{\partial}{\partial x_n}\end{pmatrix}$ be the gradient operator in $\mathbb{R}^n$ $(n>1)$, and $B_0=1$. Then, as we note in Example 9.6, $\Delta = P^*P$ is (minus) the usual Laplace operator in $\mathbb{R}^n$, and $\tau(f)=f$, and $\nu(Pf)=\frac{\partial f}{\partial\nu}$. We take as S a piece of the hypersurface $\{x_n=0\}$, and fix, on a neighbourhood of O, some non-constant harmonic function f which does not depend on the variable x_n. If the Cauchy data on S are given by means of the restriction $f_{|S}$ then the Green integral can be constructed by the formula $\mathcal{G}(\widetilde{f})(x) = \int_S \frac{\partial}{\partial\nu} g(x-.)f ds$, where $g(x-y)$ is the standard fundamental solution of convolution type of the Laplace operator in $\mathbb{R}^n$. In other words, $\mathcal{G}(\widetilde{f})$ is (minus) the potential of a double layer with density f supported on S. From the theorems on the jump of this integral and its normal derivate, we have $\mathcal{G}(\widetilde{f})^- - \mathcal{G}(\widetilde{f})^+ = f$, and $\frac{\partial}{\partial\nu}\mathcal{G}(\widetilde{f})^- - \frac{\partial}{\partial\nu}\mathcal{G}(\widetilde{f})^+ = 0$ on S. Moreover

$\frac{\partial f}{\partial \nu} = 0$ on S. Therefore Lemma 9.4 implies that the function $(\mathcal{G}(\widetilde{f}) - f)$ extends harmonically from O^+ to the whole domain O (by means of $\mathcal{G}(\widetilde{f})^-$ on O^-). This means that we can conclude the same for the integral $\mathcal{G}(\widetilde{f})^+$. However $f_{|S}$ may be the restriction of a non-constant function in D. □

At the same time, if $S = \partial D$ then the condition "$P_b(\oplus \widetilde{f_j}) = 0$ on an open subset of S" in Theorem 10.3 is not necessary (see Karepov and Tarkhanov [20]).

Corollary 10.5 (the Cartan-Kähler theorem). *Suppose that the hypersurface S, the coefficients of the operators B_j $(0 \le j \le p-1)$ in a neighbourhood of ∂D and the sections $f_j \in D'(G_{j|S})$ $(0 \le j \le p-1)$ are real analytic. Then, if $P_b(\oplus f_j) = 0$ on S, there is a section f satisfying $Pf = 0$ in some neighbourhood of S and such that $B_j f = f_j$ $(0 \le j \le p-1)$ on S.*

Proof. In view of the uniqueness theorem for solutions of $Pf = 0$ it is sufficient to find for each point $x^0 \in S$ a neighbourhood $V = V(x^0)$ on X and a solution $f \in S(V)$ such that $B_j f = f_j$ $(0 \le j \le p-1)$ on $S \cap V$. Therefore we can at once consider that the sections f_j $(0 \le j \le p-1)$ are real analytic in a neighbourhood of the compact S. Then we can construct the Green integral by the formula

$$\mathcal{G}(\widetilde{f})(x) = \int_S < C_j \Phi(x,.), f_j >_y ds \quad (x \notin S).$$

The condition of the corollary implies that the integral $\mathcal{G}(\widetilde{f})$ is a real analytic (vector-) function up to S on each sides of this hypersurface. This means that each of the integrals $\mathcal{G}(\widetilde{f}^{\pm})$ extends as a solution of the system $\Delta f = 0$ to some neighbourhood of S. If we keep the same notations for these extensions then the difference $f = \mathcal{G}(\widetilde{f})^+ - \mathcal{G}(\widetilde{f})^-$ is the solution we sought. □

æ

§11. A solvability criterion for the Cauchy problem for systems with injective symbols in the language of space bases with double orthogonality

Theorem 10.3 has been formulated so that the application of the theory of §1 (see part 1) is suggested. For this assume in addition that $q = 2$.

So, in this section we consider the solvability aspect of Problem 10.1.

Problem 11.1. *Under what conditions on the sections $f_j \in W^{s-b_j-1/2,2}(G_{j|\overline{S}})$ $(0 \le j \le p-1)$ is there a solution $f \in S(D) \cap W^{s,2}(E_{|D})$ such that $B_j f = f_j$ $(0 \le j \le p-1)$ on S ?*

Let Ω be some relatively compact subdomain of O^+.

Since $\Omega \Subset O^+$, the restriction to Ω of the Green integral $\mathcal{G}(\widetilde{f})$ defined by equality (10.1) belongs to the space $S(\Omega)_\Delta \cap W^{s,2}(E_{|\Omega})$. Hence the extendibility condition for $\mathcal{G}(\widetilde{f})$ from O^+ to the whole domain O (as a solution in the class $S_\Delta(O) \cap W^{s,2}(E_{|O})$ could be obtained by the use of a suitable system $\{b_\nu\}$ in $S_\Delta(O) \cap W^{s,2}(E_{|O})$ with the double orthogonality property. More exactly, it is required that $\{b_\nu\}$ should be an orthonormal basis in $\Sigma_1 = S_\Delta(O) \cap W^{s,2}(E_{|O})$ and an orthogonal basis in $\Sigma_2 = S(\Omega)_\Delta \cap W^{s,2}(E_{|\Omega})$.

Since $\Delta = P^*P$ is an elliptic differential operator with real analytic coefficients on X, Theorem 6.5 guarantees existence of such a basis $\{b_\nu\}$, at least if the boundary of Ω is regular (see §6). As we did in §6, for an element $\mathcal{F} \in \Sigma_1$ we shall denote by $c_\nu(\mathcal{F})$ $(\nu = 1, 2, ...)$ its Fourier coefficients with respect to the orthonormal system $\{b_\nu\}$ in Σ_1, that is, $c_\nu(\mathcal{F}) = (\mathcal{F}, b_\nu)_{H_1}$. And for an element $\mathcal{F} \in \ Sigma_2$ we shall denote by $k_\nu(\mathcal{F})$ $(\nu = 1, 2, ...)$ its Fourier coefficients with respect to the orthogonal system $\{Tb_\nu\}$ in Σ_2, that is, $k_\nu(\mathcal{F}) = \frac{(\mathcal{F}, Tb_\nu)_{H_2}}{(Tb_\nu, Tb_\nu)_{H_2}}$.

We formulate now the solvability conditions for Problem 11.1. Let $\mathcal{G}\widetilde{f}$ be the Green integral (see (10.1) constructed with "initial" data of the problem. As we noted, the restriction of the section $\mathcal{G}\widetilde{f}$ to Ω belongs to the space Σ_2.

LEMMA 11.2. *For* $\nu = 1, 2, ...$

$$k_\nu(\mathcal{G}\widetilde{f}) = \int_{\partial D} \sum_{j=0}^{p-1} < C_j k_\nu(\Phi(., y)), \widetilde{f}_j >_y ds. \tag{11.1}$$

PROOF. This consists of direct calculations with the use of equality (10.1). □

In order to determine the coefficients $k_\nu(\mathcal{G}\widetilde{f})$ $(\nu = 1, 2, ...)$ it is not necessary to know the basis $\{Tb_\nu\}$ in Σ_2. It is sufficient only to know the coefficients of the decomposition of the fundamental matrix $(\Phi(., y)$ $(y \in \partial D)$ with respect to this series. The properties of the coefficients $k_\nu(\Phi(., y) \in C^\infty_{loc}(F^*_{|X\setminus\Omega})$ we shall discuss in §12.

THEOREM 11.3. *If the boundary of the domain D is sufficiently smooth then for the solvability of Problem 11.1 it is necessary and sufficient that*

(1) $\sum_{\nu=1}^\infty |k_\nu(\mathcal{G}\widetilde{f})|^2 < \infty$;
(2) $P_b(\oplus f_j) = 0$ *in a neighborhood of some point x^0 on S.*

PROOF. The statement follows from Theorem 10.3 as Theorem 6.8 follows from Theorem 5.2. □

In conclusion we consider an example.

EXAMPLE 11.4. Aizenberg and Kytmanov [3] studied the Cauchy problem for holomorphic functions of several variables, that is, in the case $P = \begin{pmatrix} \frac{\partial}{\partial \overline{z}_1} \\ \cdots \\ \frac{\partial}{\partial \overline{z}_n} \end{pmatrix}$ and $B_0 = 1$.

In complex analysis such a problem is called the analytic extension problem for a boundary subset. They took as O the ball B with the centre at zero divided into 2 parts by means of a smooth hypersurface $S \subset B\backslash\{0\}$, and denoted by D that part of this ball which does not contain zero. A system of homogeneous harmonic polynomials $\{h_\nu^{(i)}\}$ whose restriction to the unit sphere is an orthonormal basis in $L^2(\{|x| = 1\})$ is also an orthogonal basis in the space of harmonic square- summable functions in an arbitrary ball with centre at zero. Having chosen as Ω a sufficiently small ball with centre at zero and such that $\Omega \Subset O^+$ we get a simple example of a basis with double orthogonality in Σ_1. If we solve the Cauchy problem in the class $L^2(D)$, with "initial datum" $f_0 \in L^2(S)$ then the Green integral can be constructed by the formula $\mathcal{G}(\widetilde{f})(z) = \int_S \mathcal{U}(z,.)f_0$, where $\mathcal{U}(z,.)$ is the Bochner - Martinelli kernel. Then Theorem 11.3 gives the result of Aizenberg and Kytmanov [3] with small modifications. □

We shall consider in §13 a more general range of problems. æ

§12. Carleman's formula

In this section we consider the regularization aspect of Problem 10.1.

Problem 12.1. *It is required to find a solution $f \in S(D) \cap W^{s,2}(E_{|D})$ using known values $B_j f \in W^{s-b_j-1/2,2}(G_{j|\overline{S}})$ $(0 \le j \le p-1)$ on S.*

It is easy to see from Corollary 1.8 that side by side the solvability conditions for Problem 5.1 ($q = 2$) bases with double orthogonality give the possibility to obtain a suitable formula (of Carleman) for the regularization of solutions. We shall illustrate this on example of Problem 7.1.

Let $\{b_\nu\}$ be the basis with double orthogonality, used in the previous section, in the space $(\Sigma_1 =)S(O) \cap W^{s,2}(E_{|O})$ such that the restriction of $\{b_\nu\}$ to Ω (that is, $\{Tb_\nu\}$) is an orthogonal basis of $(\Sigma_2 =)S(\Omega) \cap W^{s,2}(E_{|\Omega})$.

As above, we denote by $\{k_\nu(\Phi(.,y))\}$ the sequence of Fourier coefficients for the fundamental matrix $\Phi(.,y)$ $(y \in \Omega)$ with respect to the system $\{Tb_\nu\}$, i.e.,

$$(12.1) \qquad k_\nu(\Phi(.,y)) = \frac{1}{\lambda_\nu} \sum_{|\alpha| \le s} \int_\Omega < *D^\alpha b_\nu, D^\alpha \Phi(.,y) >_y dv \quad (\nu = 1, 2...)$$

Lemma 12.2. *The sections $k_\nu(\Phi(.,y))$ $(\nu = 1, 2...)$ are continuous, together with their derivatives up to order $(p - s - 1)$, on the whole set X.*

Proof. See part 1, Lemma 7.2. □

Using formula (12.1) one can see that the sections $k_\nu(\Phi(.,y))$ $(\nu = 1, 2...)$ extend to the boundary of Ω from each side as infinitely differentiable sections (at least, if the boundary is smooth).

LEMMA 7.3. *For any number* $\nu = 1, 2, ...$ *we have* $P' k_\nu(\Phi(., y)) = 0$ *everywhere in* $X \backslash \overline{\Omega}$.

PROOF. See part 1, Lemma 7.3. □

We consider the following kernels $\mathfrak{C}^{(N)}(x, y)$ defined for $(x, y) \in O \times X$ $(x \neq y)$:

$$\mathfrak{C}^{(N)}(x, y) = \Phi(x, y) - \sum_{\nu=1}^{N} b_\nu(x) \otimes k_\nu(\Phi(., y)) \quad (N = 1, 2, ...). \tag{12.2}$$

LEMMA 12.4. *For any number* $N = 1, 2, ...$ *the kernels* $\mathfrak{C}^{(N)} \in C_{loc}(E \boxtimes F)$ *satisfy* $P(x)\mathfrak{C}^{(N)}(x, y) = 0$ *for* $x \in O$, *and* $P'(y)\mathfrak{C}^{(N)}(x, y) = 0$ *for* $y \in X \backslash \Omega$ *everywhere except the diagonal* $\{x = y\}$.

PROOF. Since $\{b_\nu\} \subset S_\Delta(O)$, this immediately follows from Lemma 12.3. □

From the following lemma one can see that the sequence of kernels $\{\mathfrak{C}^{(N)}\}$, suitably, for example in a piece-constant way, interpolated to real values $N \geq 0$, provides a special Carleman function for Problem 12.1 (see Tarkhanov [63], §25).

LEMMA 12.5. *For any multi-index* α, $D^\alpha_y \mathfrak{C}^{(N)}(., y) \to 0$ *in the norm of* $W^{s,2}(E \otimes F^*_{y|O})$ *uniformly with respect to* y *on compact subsets of* $X \backslash \overline{O}$, *and even* $X \backslash O$ *if* $|\alpha| < p - s - n/2$.

PROOF. See part 1, Lemma 7.5. □

We can formulate now the main result of the section. For $f \in S(D) \cap W^{s,2}(E_{|D})$ we denote by $\widetilde{f} \in W^{s-b_j-1/2,2}(G_{j|\partial D})$ $(0 \leq j \leq p-1)$ an (arbitrary) extension of the section $B_j f$ from S to the whole boundary.

THEOREM 12.6 (CARLEMAN'S FORMULA). *For any solution* $f \in S(D) \cap W^{s,2}(E_{|D})$ *the following formula holds:*

$$f(x) = - \lim_{N \to \infty} \int_{\partial D} < C_j \mathfrak{C}^{(N)}(x, .), \widetilde{f}_j >_y ds \quad (x \in D). \tag{12.3}$$

PROOF. This follows from Theorems 10.3 and 11.8 as Theorem 7.6 follows from Theorems 5.2 and 6.8. □

We emphasize that the integral on the right hand side of formula (12.3) depends only on the values of the expressions $B_j f$ $(0 \leq j \leq p-1)$ on S. Thus this formula is a quantitative expression of (uniqueness) Theorem 2.8. However this gives much more than the uniqueness theorem because there is sufficiently complete information about the Carleman function $\mathfrak{C}^{(N)}$.

For holomorphic functions of several variables the Carleman formula (12.3) is first met, apparently, in [51].

Remark 12.7. The series $\sum_{\nu=1}^{\infty} k_\nu(\mathcal{G}\widetilde{f})b_\nu$ (defining the solution $\mathcal{F}$) converges in the norm of the space $W^{s,2}(E_{|O})$. The Stieltjes-Vitali theorem (see Hormander [16], 4.4.2) implies now that it converges together with all its derivatives on compact subsets of O. Then, as in §7, one can see that the limit in (12.3) is reached in the topology of the space $C^{\infty}_{loc}(E_{|O})$.

æ

§13. Examples for systems of the simplest type

In this section we extend the results of 18 to overdetermined systems of the simplest type.

We suppose that P is a (overdetermined) differential operator of the simplest type in $\mathbb{R}^n$ (see §8). Let $O = B_R$ be the ball in $\mathbb{R}^n$ with centre at zero and radius $0 < R < \infty$, and S be a smooth closed hypersurface in B_R dividing this ball into 2 connected components O^+, and $D = O^-$ so that the domain O^+ contains zero. We consider the following problem (of Cauchy).

Problem 13.1. *Let $f_0 \in C_{loc}(E_{|S})$ be a summable section of E on S. It is required to find a solution $f \in S(D) \cap C_{loc}(E_{|D\cup S})$ such that $f_{|S} = f_0$.*

As the fundamental solution of the differential operator P we can take the matrix $\Phi(x,y) = P'(y)g(x-y)$, where $g(x-y)$ is the standard fundamental solution of convolution type of the Laplace operator in $\mathbb{R}^n$ with the opposite sign. Then the Green integral (5.1) is written in the following form:

$$\mathcal{G}\widetilde{f}(x) = \frac{1}{\sqrt{-1}} \int_S \Phi(x,.)\sigma(P)(\nu) f_0 ds \ (x \notin S).$$

It is easy to see from the structure of the fundamental matrix Φ that the components of the section $\mathcal{G}\widetilde{f}$ are harmonic functions everywhere in B_R (and even in $\mathbb{R}^n$) except on the set S.

To obtain a solvability criterion for Problem 13.1 we can use the basis with double orthogonality constructed in Lemma 8.4.

Our principal result will be formulated in the language of the coefficients

$$k_\nu^{(i)} = \frac{1}{\sqrt{-1}} \int_S P^{*'}(y)[\frac{1}{n+2\nu-2}\frac{\overline{h_\nu^{(i)}(y)}}{|y|^{n+2\nu-2}}]\sigma(P)(\nu) f_0 ds \ (\nu = 1,2,...).$$

Theorem 13.2. *For solvability of Problem 13.2, it is necessary and sufficient that*

(1) $\limsup_{\nu\to\infty} \max_{1\le i\le J(\nu)} \sqrt[\nu]{|k_\nu^{(i)}(y)|} \le \frac{1}{R}$;
(2) $P_b f_0 = 0$ *in a neighborhood of some point x^0 on S.*

PROOF. The statement follows from Theorem 10.3 as Theorem 8.6 follows from Theorem 5.2. □

In conclusion we give the corresponding variant of Carleman's formula. For each number $N = 1, 2...$ we consider the kernel $\mathfrak{C}^{(N)}(x, y)$ defined, for all $y \neq 0$ off the diagonal $\{x = y\}$, by the equality

$$\mathfrak{C}^{(N)}(x,y) = \Phi(x,y) - \Phi(0,y) + \sum_{\nu=1}^{\infty} \sum_{i=1}^{J(\nu)} h_\nu^{(i)}(x) P^{*'}(y) [\frac{1}{n+2\nu-2} \frac{\overline{h_\nu^{(i)}(y)}}{|y|^{n+2\nu-2}}].$$

LEMMA 13.3. *For any number $N = 1, 2, ...$, the kernel $\mathfrak{C}^{(N)}$ is an infinitely differentiable section of $E \boxtimes F$, harmonic with respect to x, and satisfying $P'(y)\mathfrak{C}^{(N)}(x, y) = 0$ for all $y \neq 0$ off the diagonal $\{x = y\}$.*

PROOF. This follows from the properties of the matrix Φ and the polynomials $h_\nu^{(i)}(y)$. □

We note that since $\mathfrak{C}^{(N)}$ is a "remainder" summand in the formula (8.2), $\mathfrak{C}^{N)}(x, y) \to 0$ $(N \to \infty)$, together with all its derivatives uniformly on compact subsets of the cone $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |y| > |x|\}$.

THEOREM 13.4 (CARLEMAN'S FORMULA). *For any solution $f \in S(D) \cap C_{loc}(E_{|D \cup S})$ whose restriction to S is summable there, the following formula holds*

$$f(x) = -\frac{1}{\sqrt{-1}} \lim_{N\to\infty} \int_S \mathfrak{C}^{(N)}(x, .)\sigma(P)(\nu) f_0 ds \quad (x \in D). \tag{13.1}$$

PROOF. This is similar to the proof of Theorem 12.6. □

Remark 13.5. As in Theorem 12.6, the convergence in (13.1) is uniform on compact subsets of the domain D together with all the derivatives.

æ æ

References

1. L.A. Aizenberg, *Carleman's formulas in complex analysis. First Applications.*, Nauka, Novosibirsk, 1990, pp. 248 p.; English transl. in Kluwer Ac. Publ, Dordrecht, 1993.
2. L.A. Aizenberg, *Simple conditions for holomorphic continuation from a part of the boundary of a convex domain to the whole domain.*, Prepr. Royal Inst. of Techn., Trita, Mat., Stockholm (1991 15), 7 pp..
3. L.A. Aizenberg, Kytmanov A.M., *On the possibility of holomorphic extension into a domain of functions defined on a connected piece of its boundary*, Mat. Sbornik **182 4.** (1991.), 490–507; English transl. in Math. USSR Sbornik, **72** (1992 2), 467–483..
4. Aronszajn N., *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68** (1950), 337–404..
5. Berezanskii Ju. M., *Expansions in eigenfunctions of selfadjoint operators.*, Naukova dumka, Kiev, 1965, pp. 800p; English transl. in Providence, AMS, 1968.
6. Bergman S., *The kernel function and conformal mapping: Second (revised) edition. (Mathematical Surveys, V)*, AMS, 1970..
7. Bungart L., *Boundary kernel functions for domains on complex manifolds*, Pacif. J. Math. **14** (1964 4), 1151–1164.
8. Carleman T., *Les fonctions quasianalytiques.*, Gauthier-Villars, Paris, 1926, pp. 115 p..
9. Chirka E.M., *Analytic representation of CR-functions*, Mat. sbornik **98** (1975 4), 591–623;; English transl. in Math. USSR Sbornik **27** (1975 4), 526–553.
10. Dudnikov P.I. and Samborskii S.N., *Boundary value and initial-boundary value problem for linear overdetermined systems of partial differential equations*, Results of Sciences and Technics, Modern Problems of Mathematics. Fundamental Trends, VINITI AN SSSR **65** (1991), 5–93. (in Russian)

11 Eskin G.I., *Boundary value problem for elliptic pseudo- differential equations*, Nauka, Moscow, 1973, pp. 232 p.; English transl. in Prov., AMS, 1981.

12. Fok V.A., Kuny F.M., *On the introduction of an "annihilating" function in the dispersion equations for gases*, Dokl. AN SSSR. **127** (1959 6), 1195–1198. (in Russian)
13. Fursikov A.V., *The Cauchy problem for elliptic equations of the second order in a conditionally -correct formulating*, Trudy Mosk. matem. ob-va **52** (1990), 138–174. (in Russian)
14. Hadamard J., *Le problème de Cauchy et les equations aux derivées partielles linéares hyperboliques*, Gauthier - Villars, Paris, 1932.
15. G.M Henkin, *Method of integral representations in complex analysis*, Results of Sciences and Technics, Modern Problems of Mathematics. Fundamental Trends, VINITI AN SSSR **7** (1985), 23–124. (in Russian)
16. L. Hörmander, *The analysis of linear partial differential operators. Distribution theory and Fourier analysis.* I, Springer- Verlag, Berlin - Heidelberg, 1983; *The analysis of linear partial differential operators. Differential operators with constant coefficients.* II, Springer - Verlag, Berlin - Heidelberg, 1983; *The analysis of linear partial differential operators. Pseudo - differential operators.* III, Springer - Verlag, Berlin - Heidelberg, 1985; *The analysis of linear partial differential operators. Fourier integral operators.* IV, Springer - Verlag, Berlin - Heidelberg, 1985.
17. Ivanov V.K., *Obratnaya zadacha potentiala dlya tela, blizkogo k dannomu*, Izvestiya AN SSSR. Ser. Mat. **20** (1956 6). (in Russian)
18. Jarmuhamedov S.J., *On the Cauchy problem for Laplace's equation*, Dokl AN SSSR **235** (1977 2), 281–283; English transl. in Sov. Math. Dokl. **18** (1977 4), 939–943.
19. Karepov O.V. and Tarkhanov N.N., *On the Cauchy problem for holomorphic functions of Hardy class H^2.*, Prepr. Inst. of Physics, Academy of sciences, Siberian branch, Krasnoyarsk (1990 53M), 21 p.
20. Karepov O.V. and Tarkhanov N.N. The Fisher-Riesz method in the ill-posed Cauchy problem for systems with injective symbols, Dokl. Ross. AN **326** (1992 5), 776–780; English transl. in Russian Ac. Sci. Dokl. Math. **46** (1993 2), 354–358.

21. Karepov O.V., *On holomorphic extension of functions from subsets of the Shilov boundary of a circular starlike domain*, Siberian Math. Journ. (1995) (to appear).
22. Kondrat'ev V.A. and Landis E.M., *Qualitative theory for linear differential equations of the second order*, Results of Sciences and Technics, Modern Problems of Mathematics. Fundamental Trends, VINITI AN SSSR **32** (1988), 99–215. (in Russian)
23. Koppelman W. and Pincus J.D., *Spectral representations for a finite Hilbert transformation*, Math. Z. **71** (1959 4), 399–407.
24. Koroljuk T.N., *On the Cauchy problem for Laplace's equation*, Izvestija VUZ. Mathematika 3 (1973), 53–55. (in Russian)
25. Krasichkov I.F., *Systems of functions with the dual orthogonality property*, Mat. Zametki **4** (1968 5), 551–556; English Transl. in Math. Notes, **4** (1968 5), 821–824.
26. Krein M.G and Nudel'man P.J., *On some new problems for Hardy class functions and continuous families of functions with double orthogonality*, Dokl. AN SSSR **209** (1973 3), 537–540; English Transl. in Sov. Math. Dokl. **14** (1973 2), 435–439.
27. Kudrjavcev L.D. and Nikol'skii S.M., *Spaces of differentiable functions of several variables and embedding theorems*, Results of Sciences and Technics, Modern Problems of Mathematics. Fundamental Trends, VINITI AN SSSR **26** (1987), 5–157. (in Russian)
28. Kytmanov A.M. and Tarkhanov N.N., *Boundary properties of the Bochner-Martinelli integral (review of results)*, Prepr. Inst. of Physics, Academy of sciences, Siberian branch, Krasnoyarsk (1990 54M), 50 p.
29. Landau H.J.,Pollak H.O., *Prolate spheroidal wave functions, Fourier analysis and uncertainty. II.*, Bell. Syst. Techn. J. **40** (1961 1), 65–86.
30. Landau H.J. and Pollak H.O., *Prolate spheroidal wave functions. III. The dimension of the space of essentially time- and band-limited signals*, Bell. Syst. Techn. J. **41** (1962 4), 1295–1336.
31. Landis E.M., *On some properties of solution of elliptic equations*, Dokl. AN SSSR **107** (1956 5), 640–643. (in Russian)
32. Lavrent'ev M.M., *On the Cauchy problem for Laplace's equation*, Izvestija AN SSSR. Ser. mat. **20** (1956), 819-842. ((in Russian))
33. Lavrent'ev M.M., *On the Cauchy problem for linear elliptic equations of the second order*, Dokl. AN SSSR **112** (1957 2), 195–197. (in Russian)
34. Lavrent'ev M.M., *On some ill-posed problems of mathematical physics*, Academy of sciences of USSR, Siberian branch, Centre of calculations, Novosibirsk, 1962, pp. 92p..
35. J.L. Lions and E. Magenes, *Problems aux limites non homogenes at applications. I-III*, Dunod, Paris, 1966–1968.
36. Mahmudov O.I., *The Cauchy problem for a system of equations of elasticity theory in Eucledian space*, Diss. ... Kand. Fiz.-mat. nauk. Samarkand, 1990, pp. 80p.. (in Russian)
37. Maz'ya V.G. and Havin V.P., *On the solutions of the Cauchy problem for Laplace's equation (uniqueness, normality, approximation)*, Trudy Mosk. Mat. Ob-va **307** (1974), 61–114; English Transl. in Transactions of the Moscow Math. Soc. **30** (1974), 65–118.
38. Mergeljan S.N., *Harmonic approximation and approximate solution of the Cauchy problem for Lapla- ce's equation*, Uspechi. Mat. Nauk **II** (1956 vypusk.5), 3–26. (in Russian)
39. Mikhailov V.P., *Partial differential equations*, Nauka, Moscow, 1976, pp. 392 p.. (in Russian)
40. Nacinovich M., *Cauchy problem for overdetermined systems*, Ann. di Mat. Pura ed Appl.(IV) **156** (1990), 265–321..
41. Newman D.J., *Numerical method for solution of an elliptic Cauchy problem*, J. Math. and Phys. **5** (1960 1), 72–75.
42. Patil D.I., *Representation of H^p -functions jour Bull. Amer. Math. Soc.* **78** (1972 4), 617–620.
43. Privalov I.I. and Kuznecov I.P., *Boundary problems and various classes of harmonic functions defined for arbitrary domains*, Matem. sbornik **6** (1939 3), 345–376. (in Russian)

44. Pucci C., *Discussione del problema di Cauchy pur le equazioni di tipo ellittico*, Ann. Mat. Pura ed Appl. **46** (1958), 131–153.
45. S. Rempel and B.-W. Shulze, *Index theory of elliptic boundary problems*, Akademie-Verlag, Berlin, 1982.
46. Riesz F., Sz.-Nagy B., *Lecons d'analyse fonctionelle: Sixieme edition*, Academiai Kiado, Budapest, 1972.
47. Rojtberg J.A., *On boundary values of generalized solutions of elliptic equations*, Mat. sb. **86 (128)** (1971 2), 248–267; English Transl. in Math. USSR Sbornik **15** (1971 2), 241–260.
48. Samborskii S.N., *Coercive boundary-value problems for overdetermined systems (elliptic problems)*, Ukrainian Math. journ. **36** (1984 3), 340–346.
49. Shapiro H.S., *Stefan Bergman s theory of doubly- orthogonal functions. - An operator-theoretic approach*, Proc. Roy. Acad. Sect. **79 6** (1979), 49–56.
50. Shapiro H.S., *Reconstructing a function from its values on subset of its domain. - A Hilbert space approach*, Journal of approximation theory **46** (1986 4), 385–402.
51. A.A. Shlapunov and N.N. Tarkhanov, *On the Cauchy problem for holomorphic functions of Lebesgue class L^2 in domain*, Siberian math. journal **33** (1992 5), 914–922.
52. A.A. Shlapunov and N.N. Tarkhanov, *Bases with double orthogonality in the Cauchy problem for systems with injective symbol. Elliptic systems*, Prepr. Inst. of Physics, Academy of sciences, Siberian branch, Krasnoyarsk (1990 56M), 54 p..
53. A.A. Shlapunov and N.N. Tarkhanov, *Bases with double orthogonality in the Cauchy problem for systems with injective symbol. General case*, Prepr. Inst. of Physics, Academy of sciences, Siberian branch, Krasnoyarsk (1992 57M), 30 p..
54. Shlapunov A.A. and Tarkhanov N.N., *Bases with double orthogonality in the Cauchy problem for systems with injective symbol*, Dokl. Ross. AN **326** (1992 1), 45–49; English Transl. in Russian Ac. Sci. Dokl. Math. **46** (1993 2), 225–230.
55. Shlapunov A.A., *On the Cauchy problem for Laplace's operator*, Siberian Math. Journ **33** (1992 3), 205–215.
56. Slepian D. and Pollak H.O., *Prolate spheroidal wave functions, Fourier analysis and uncertainty. I.*, Bell. Syst. Techn. J. **40** (1961 1), 43–63.
57. Slepian D., *Prolate spheroidal wave functions, Fourier analysis and uncertainty. IV. Extension to many dimensions; generalized prolate spheroidal functions*, Bell. Syst. Techn. J. **43** (1964 6), 3009-3057.
58. Stein E., *Singular integrals and differentiability properties of functions*, Princeton University Press,, Princeton, 1970.
59. Steiner A., *Abschnitte von Randfunktionen beschränkter analytischer Funktionen*, Topics in Analysis - Proceedings 1970 (O.E. Letho, I.S. Louhivaara, and R.H. Nevanlinna., eds.), Lect. Notes Math. (419), 1974, pp. 342–351.
60. Straube E.J., *Harmonic and analytic functions admitting a distribution boundary value*, Ann. Sc. norm. super. Pisa cl. sci. **11** (1984 4), 559–591.
61. Tarkhanov N.N., *On the Carleman matrix for elliptic systems*, Dokl. AN SSSR **284** (1985 2), 294–297; English Transl. in Sov. Math. Dokl **. 32** (1985 2), 429–432.
62. Tarkhanov N.N., *Solvability criterion for the ill-posed Cauchy problem for elliptic systems*, Dokl. AN SSSR **308** (1989 3), 531–534; English Transl. in Sov. Math. Dokl **. 40** (1990 2), 341–345.
63. N.N. Tarkhanov, *Laurent series for solutions of elliptic systems*, Nauka, Novosibirsk, 1991, pp. 318 p.. (in Russian)
64. Tarkhanov N.N., *Fourier series in the Cauchy problem for solutions of elliptic systems*, Non classic and ill-posed problems of mathematical physics and analysis (Abstracts of soviet - italian symposium, 2-6 okt. 1990) (1990), izd-vo SamGU, Samarkand, 40.
65. N.N. Tarkhanov, *The parametrix method in the theory of differential complexes*, Nauka, Novosibirsk, 1990, pp. 247 p.. (in Russian)

66. Zin G., *Esistenza e rappresentazione di funzioni analitiche, le quali, su una curva di Jordan, si riducono a una funzione assegnata*, Ann. Mat. Pura ed Appl. **34** (1953), 365–405.
67. Znamenskaya L.N., *Criterion for holomorphic continuability of the functions of class L defined on a portion of the Shilov boundary of a circular strongly starlike domain*, Siberian Math. journ **31** (1990 5), 175–177.

æ