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ON THE CAUCHY PROBLEM FOR OPERATORS WITH INJECTIVE SYMBOLS IN THE SPACES OF DISTRIBUTIONS

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Abstract. Let D be a bounded domain in n-dimensional Eucledian space ($n \geq 2$) having smooth boundary ∂D. We indicate appropriate Sobolev spaces with negative smoothness in D in order to consider the non-homogeneous ill-posed Cauchy problem for an overdetermined operator A with injective symbol. We prove that elements of the indicated Sobolev spaces have traces on the boundary. This easily leads to a weak formulation of the Cauchy problem and to the corresponding Uniqueness Theorem. We also describe solvability conditions of the problem and construct its exact and approximate solutions. Namely, we obtain Carleman formula recovering a vector-function u from the indicated negative Sobolev class via its Cauchy data on an open connected set $\Gamma \subset \partial D$ and values of Au on the domain D. Some instructive examples are considered.

Keywords. Ill-posed Cauchy Problem, elliptic operators, Carleman's formula.

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Introduction

The study of the Cauchy problem for elliptic equations is going on since 1920-th (see, for instance, [9] in Partial Differential Equations or [5] in Complex Analysis). However the essential progress in the study appeared after the high motivations related to the needs of applications only (cf. [10], [11], [8]). Actually, the Cauchy problem naturally arises in Hydrodynamics, in Electrodynamics, in Geophysics, in Elasticity Theory, in Theoretical Physics, in Theory of Signal Transmission and so on (see, for instance, [12], [22]). The problem was actively studied from various points of view (uniqueness, solvability, regularization, stability) since the middle of the last century (see, for instance, bibliographies to [12], [2], [22] and [4]). We present an approach originated from Complex Analysis [3] and developed for the

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homogeneous Cauchy problem for determined and overdetermined elliptic systems with real analytic coefficients in [18].

In contrast to [18] we consider the *non-homogeneous* Cauchy problem. Of course, these problems (the *homogeneous* and *non-homogeneous* ones) are equivalent in many spaces if the symbol of the corresponding differential operator is invertible. However we study, in general, overdetermined elliptic system, say A (a typical example is the Maxwell system in stationary situation). Then the Cauchy problems are equivalent if and only there is information on the solvability of the operator equation $Au = f$ in proper function spaces. Even for operators with real analytic coefficients we have information on *local* solvability of the operator equation only; moreover, in general, there is no such information for operators with smooth coefficients (see [21]). The second advantage of our results consists in replacing the real analyticity of the coefficients by a more weak *uniqueness condition*. Finally, we obtained the solvability conditions and constructed Carleman formulas for the Cauchy data from a much more wide class of distributions than [18]. Actually, we follow [15] (cf. also [14], [22])) in order to correctly define traces for elements of spaces of distributions of finite orders of singularity.

1 Differential operators and Sobolev spaces

Let X be a C^∞-smooth manifold of dimension $n \geq 2$ with smooth boundary ∂X; we assume that it is enclosed into a smooth manifold $\tilde{X}$ (without boundary) of the same dimension. For smooth $\mathbb{C}$-vector bundles E and F of ranks k and l respectively over X we denote $\mathrm{Diff}_m(X; E \to F)$ the space of all the linear differential operators between the bundles E and F of orders which are less or equal than m. Then, for any open set $O \subset X$ over which the manifold and the bundles are trivial, sections of the bundles may be interpreted as (vector-) functions and an operator $A \in \mathrm{Diff}_m(X; E \to F)$ is given by an $(l \times k)$-matrix of scalar differential operators with smooth coefficients in O.

Denote by E^* the adjoint bundle for E. Any Hermitian metrics $(.,.)_x$ in layers of E induces a (sesquilinear) bundle isomorphism $\star_E : E \to E^*$ given by $\langle \star_E v, u\rangle_x = (u, v)_x$ for all sections u and v of the bundle E; here $\langle .,. \rangle_x$ is the natural pairing between layers of E^* and E. Fix a volume form dx on X, identifying the dual and the adjoint bundles. For $A \in \mathrm{Diff}_m(X; E \to F)$ denote $A' \in \mathrm{Diff}_m(X; F^* \to E^*)$ and $A^* \in \mathrm{Diff}_m(X; F \to E)$ the transposed and formally adjoint operators respectively. Obviously, $A^* = \star_E^{-1} A' \star_F$, see [22, 4.1.4].

Let $\sigma(A)$ be (homogeneous) symbol of order m of the operator A, living on (real) cotangent bundle T^*X of the manifold X. We always assume that $\sigma(A)$ is injective away from zero section of the bundle T^*X. We will say that A is *elliptic* if $\operatorname{rank} E = \operatorname{rank} F$ and that A is *overdetermined elliptic* otherwise. The operator $\Delta = A^*A \in \mathrm{Diff}_{2m}(X; E \to E)$ is usually called the generalized Laplacian for A; easily it is a strongly elliptic operator of order $2m$ on X.

Let $\overset{\circ}{X}$ be the interior of the manifold X, and D be bounded domain in $\overset{\circ}{X}$ with infinitely smooth boundary ∂D. Denote $C^\infty(D, E)$ the Frechét space of infinitely differentiable sections of the bundle E over D and denote $C^\infty(\overline{D}, E)$ the space of all such sections which has derivatives of any order that can be continuously extended to $\overline{D}$. For an open (in the topology of ∂D) subset $\Gamma \subset \partial D$, let $C^\infty_{comp}(D \cup \Gamma, E)$ be the set of $C^\infty(\overline{D}, E)$-sections with compact supports in $D \cup \Gamma$. Then $C^\infty_{comp}(D, E)$ corresponds to $\Gamma = \emptyset$.

For a distribution-section $u \in (C^\infty_{comp}(D, E))'$ we always understand Au in the sense of distributions in D. Denote $S_A(D)$ the set of all solutions to the equation $Au = 0$ in D. It is well known that, due to ellipticity of A, the elements of $S_A(D)$ belong to $C^\infty(D, E)$ (see, for example [21]). We say that a section $u \in S_A(D)$ has a finite order of growth near ∂D if for any $x^0 \in \partial D$ there are a ball $B(x^0, R)$ and such constants $c > 0$, $\gamma > 0$ that $|u(x)| \leq c\, dist(x, \partial D)^{-\gamma}$ for all $x \in B(x^0, R) \cap D$. The compactness of ∂D guarantees that c and γ can be chosen in such a way that this inequality holds for all $x \in \partial D$. The space of sections $u \in S_A(D)$ with finite order of growth near ∂D will be denoted $S^F_A(D)$.

Fix a Dirichlet system $\{B_j\}_{j=0}^{m-1}$ of order $(m-1)$ in a neighborhood of ∂D. More exactly, each B_j is a differential operator of type $E \to F_j$ and of order $m_j \leq m-1$, $m_j \neq m_i$ for $j \neq i$ (here F_j are smooth bundles over a neighborhood U of ∂D with ranks equal to k). Moreover, the (principal) symbol $\sigma(B_j)$ of each B_j, restricted to co-normal vectors to ∂D, has the same rank as F_j. Without loss of the generality we assume $m_j = j$. A typical example of $\{B_j\}_{j=0}^{m-1}$ is the system $\{\frac{\partial^j}{\partial \nu^j}\}_{j=0}^{m-1}$ of normal derivatives with respect to ∂D.

As the symbol of A is injective, ∂D is not characteristic for A. Hence there is a Dirichlet system $\{C_j\}_{j=0}^{m-1}$ of order $(m-1)$ in a neighborhood U of ∂D with $C_j \in \mathrm{Diff}_{m-j-1}(U; F_{|U} \to F_j)$, such that for all $g \in C^\infty(X, F)$, $v \in C^\infty(X, E)$ the (first) Green formula holds true:

$$\int_{\partial D} \sum_{j=0}^{m-1} (B_j u, C_j g)_x ds = \int_D \left((Au, g)_x - (u, A^* g)_x\right) dx. \tag{1.1}$$

where ds is the volume form on ∂D induced from X (see [22, lemma 9.2.7]).

For $u \in C^\infty(\overline{D}, E)$ we set $t(u) = \oplus_{j=0}^{m-1} B_j u$; then $t(u)$ represents the Cauchy data with respect to A. Similarly, for $g \in C^\infty(\overline{D}, F)$ we set $n(g) = \oplus_{j=0}^{m-1} C_j g$; the operator $n(g)$ represents the Cauchy data with respect to A^*.

Since we consider, in general, *overdetermined* systems, it is natural to assume that A may be included to an *elliptic* complex

$$0 \to C^\infty(E) \stackrel{A}{\to} C^\infty(F) \stackrel{A_1}{\to} C^\infty(G) \to \cdots \to 0. \tag{1.2}$$

This means that $A_1 \circ A = 0$ and the corresponding symbolic complex is exact away from zero section of the bundle T^*X. It is possible, if A is *sufficiently regular* (see, for instance, [21, §3]). Of course, operators with constant coefficients are sufficiently regular. If A is elliptic then it is regular and $A_1 \equiv 0$.

We also need the so called uniqueness condition for A in small on $\overset{\circ}{X}$.

Property 1.1 (Uniqueness Condition). If $u \in S_A(D)$ satisfies $u = 0$ on a non-empty open subset O in $D \subset \overset{\circ}{X}$ then $u \equiv 0$ in D.

We always assume that both the operators A and $A^* \oplus A_1$ have Property 1.1. Of course, it is true if all the objects in the consideration are real analytic.

We denote $L^2(D, E)$ the Hilbert space of all the measurable sections in D endowed with the scalar product $(u, v)_{L^2(D,E)} = \int_D (u, v)_x dx$. For $s \in \mathbb{N}$ we denote $H^s(D, E)$ the Sobolev space of sections of the bundle E over D having all the derivatives up to order s in $L^2(D, E)$. The Sobolev space $H^s(D, E)$ with fractional $s \in \mathbb{R}_+ \setminus \mathbb{Z}_+$ are defined with the standard interpolation (see, for instance, [22, §1.4.11]). For negative smoothness the Sobolev spaces are usually defined with the use of a proper duality. In addition to the standard scale $\tilde{H}^{-s}(D, E)$, $s \in \mathbb{N}$ (see [1]), we consider also the following two scales of spaces adopted for studying Dirichlet problem for strongly elliptic operators (cf. [15], [22, Chapters 1, 9], [14], [19]). We denote by $C^\infty_{m-1}(\overline{D}, E)$ the subspace in $C^\infty(\overline{D}, E)$ consisting of sections, vanishing up to order $m - 1$ on ∂D. For $s \in \mathbb{N}$ and $u \in C^\infty(\overline{D}, E)$ we define two types of negative norms:

$$\|u\|_{-s} = \sup_{v \in C^\infty(\overline{D},E)} \frac{|(u, v)_{L^2(D,E)}|}{\|v\|_{H^s(D,E)}}, \quad |u|_{-s} = \sup_{v \in C^\infty_{m-1}(\overline{D},E)} \frac{|(u, v)_{L^2(D,E)}|}{\|v\|_{H^s(D,E)}}.$$

It is more correctly to write $\| \cdot \|_{-s,D}$ and $| \cdot |_{-s,D}$, but we prefer to omit index D if it does not lead to misunderstandings. It is convenient to set $\| \cdot \|_{0,D} = \| \cdot \|_{L^2(D)}$.

Denote the completion of the space $C^\infty(\overline{D}, E)$ with respect to these norms $H^{-s}(D, E)$ and $H(D, E, | \cdot |_{-s})$ respectively. It is not difficult to show that $H^{-s}(D, E) = (C^\infty(\overline{D}, E))'$ and $H(D, E, | \cdot |_{-s}) = (C^\infty_{m-1}(\overline{D}, E))'$ (cf. [22, Theorem 1.4.28]). In particular, the elements of these spaces are distributions of finite orders of singularity over D and the spaces itself may be called the Sobolev spaces of negative smoothness s. Besides, the duality implies that the Banach space $H^{-s}(D,E)$, $s>0$, is a Hilbert space with the scalar product

$$(u,v)_{-s} = \frac{1}{4}\left(\|u+v\|^2_{-s} - \|u-v\|^2_{-s} + \|iu+v\|^2_{-s} - \|iu-v\|^2_{-s}\right) \qquad (1.3)$$

coherent with the norm $\|\cdot\|_{-s}$.

Obviously, $H^{-s}(D,E) \hookrightarrow H(D,E,|\cdot|_{-s}) \hookrightarrow \tilde{H}^{-s}(D,E)$, and, similarly, $H^{-s}(D,E) \hookrightarrow H^{-s-1}(D,E)$, $H(D,E,|\cdot|_{-s}) \hookrightarrow H(D,E,|\cdot|_{-s-1})$. It is clear that any element $u \in H^{-s}(D,E)$ extends up to an element $U \in H^{-s}(\overset{\circ}{X},E)$ via $\langle U,v\rangle_{\overset{\circ}{X}} = \langle u,v\rangle_D$ for all $v \in H^s(\overset{\circ}{X},E)$; here $\langle\cdot,\cdot\rangle_D$ is the pairing on $H \times H'$ for a space H of distributions over D. It is natural to denote it by $\chi_D u$ (sometimes below χ_D is also the characteristic function of D). Thus, the defined in this way linear operator $\chi_D : H^{-s}(D,E) \to H^{-s}(\overset{\circ}{X},E)$, $s \in \mathbb{Z}_+$, is obviously bounded.

Lemma 1.2. *An operator $A \in \mathrm{Diff}_m(X;E\to F)$ induces a linear bounded operator $A : H^{-s}(D,E) \to H(D,F,|\cdot|_{-s-m})$, $s \in \mathbb{Z}_+$.*

Proof. Follows from Green formula (1.1). □

However there are no reasons that $Au \in H^{-s-m}(D,F)$ for an element $u \in H^{-s}(D,E)$ and there are no reasons for elements of $H^{-s}(D,E)$ to have traces on ∂D. Thus, we introduce two more types of negative norms. We will use them to study the Cauchy problem below. Actually we follow the approach [22, §9.2, 9.3] which was realized for the Laplacian A^*A. Namely, for $s \in \mathbb{Z}_+$ we denote completions of the space $C^\infty(\overline{D},E)$ with respect to the graph-norms

$$\|u\|_{-s,A} = \left(\|u\|^2_{-s} + \|Au\|^2_{-s-m}\right)^{1/2},$$

$$\|u\|_{-s,t} = \left(\|u\|^2_{-s} + \sum_{j=0}^{m-1} \|B_j u\|^2_{-s-j-1/2,\partial D}\right)^{1/2}.$$

by $H^{-s}_A(D,E)$ and $H^{-s}_t(D,E)$ respectively. Then the differential operators A and t induce linear bounded operators $A_{-s} : H^{-s}_A(D,E) \to H^{-s-m}(D,F)$ and $t_{-s} : H^{-s}_t(D,E) \to \oplus_{j=0}^{m-1} H^{-s-j-1/2}(\partial D, F_j)$, $s \in \mathbb{Z}_+$.

Remark 1.3. The spaces $H^{-s}(D,E)$, $H^{-s}_t(D,E)$, $H(D,E,|\cdot|_{-s})$ are well-known. For instance, given distribution F over D and distributions $\oplus_{j=0}^{m-1} u_j$ over ∂D, consider the Dirichlet problem of finding a distribution u satisfying

$$\begin{cases} A^*Au &= F \quad \text{in } D, \\ t(u) &= \oplus_{j=0}^{m-1} u_j \quad \text{on } \partial D. \end{cases} \qquad (1.4)$$

(cf. [15] for scalar operators or [14]). Results of [19, theorem 2.26] imply that, under the Uniqueness Condition, Problem (1.4) is uniquely solvable on the scale of Sobolev spaces $H^s(D,E)$, $s \in \mathbb{Z}$, with the data $F \in H(D,E,|\cdot|_{s-2m})$ and $\oplus_{j=0}^{m-1} u_j \in \oplus_{j=0}^{m-1} H^{s-j-1/2}(\partial D, F_j)$. Denote $\mathcal{P}^{(D)}$ the operator mapping the data $\oplus_{j=0}^{m-1} u_j$ and $F=0$ to the unique solution of the Dirichlet problem. Similarly, denote $\mathcal{G}^{(D)}$ the operator mapping F to the unique solution of the Dirichlet problem (1.4) with zero boundary conditions. Actually, $\mathcal{G}^{(D)}$ is the Green function of the Dirichlet problem (1.4) and $\mathcal{P}^{(D)}$ is the corresponding Poisson integral. It follows from [22, theorem 9.3.17] and [19, theorem 2.26, corollary 2.31] that the operators $\mathcal{P}^{(D)}$, $\mathcal{G}^{(D)}$ are continuously act on the Sobolev spaces:

$$\mathcal{P}_s^{(D)} : \oplus_{j=0}^{m-1} H^{s-j-1/2}(\partial D, F_j) \to H^s(D,E), \quad s \geq m,$$

$$\mathcal{G}_s^{(D)} : H(D,E,|\cdot|_{s-2m}) \to H^s(D,E), \quad s \geq m,$$

$$\mathcal{P}_s^{(D)} : \oplus_{j=0}^{m-1} H^{s-j-1/2}(\partial D, F_j) \to H_t^s(D,E), \quad s < m,$$

$$\mathcal{G}_s^{(D)} : H(D,E,|\cdot|_{s-2m}) \to H_t^s(D,E), \quad s < m.$$

Moreover, they give complete solution to the Dirichlet problem in these spaces. □

Theorem 1.4. *Linear spaces $H_A^{-s}(D,E)$ and $H_t^{-s}(D,E)$, $s \in \mathbb{Z}_+$, coincide and their norms are equivalent.*

Proof. It follows from the definition that it is sufficient to check the relations between the norms on $C^\infty(\overline{D},E)$. But Green formula (1.1) easily implies that the norm $\|\cdot\|_{-s,A}$ is not stronger than the norm $\|\cdot\|_{-s,t}$:

$$\|u\|_{-s,A} \leq \left(1 + \|A^*_{m+s}\|^2 + \sum_{j=0}^{m-1} \|C_j^{m+s}\|^2\right)^{1/2} \|u\|_{-s,t}, \quad s \in \mathbb{Z}_+$$

where bounded linear operators $A^*_{m+s} : H^{m+s}(D,F) \to H^s(D,E)$ and $C_j^{m-s} : H^{m+s}(D,F) \to H^{s+j+1/2}(\partial D, F_j)$ are induced by differential operators A^* and C_j respectively. In order to continue the proof we need the following lemma.

Lemma 1.5. *For any data $\oplus_{j=0}^{m-1} g_j \in \oplus_{j=0}^{m-1} C^\infty(\partial D, F_j)$ there is a section $g \in C^\infty(\overline{D},F)$ with $n(g) = \oplus_{i=0}^{m-1} g_j$ and*

$$\|g\|_{H^{s+m}(D,F)} \leq \sum_{j=0}^{m-1} \gamma_j \|g_j\|_{H^{s+j+1/2}(\partial D, F_j)}, \quad s \in \mathbb{Z}_+, \tag{1.5}$$

with constants γ_j which do not depend on both g_j and g.

Proof. The existence of a section $\tilde{g} \in C^\infty(\overline{D}, F)$ with $n(\tilde{g}) = \oplus_{j=0}^{m-1} g_j$ is well-known (see, for instance, [22, §9]. The corresponding estimates can be proved with the use of the continuity of the Poisson integral and Green function related to the Dirichlet problem for generalized Laplacians (see remark 1.3).

Indeed, if both A and A_1 are the first order operators then the Cauchy data $n(\cdot)$ with respect to A are presented by a surjective matrix C_0 on ∂D. Then one can set $g = \mathcal{P}_1(C_0^*(C_0 C_0^*)^{-1} g_0)$, where $\mathcal{P}_1 u_0^{(1)}$ is the Poisson integral of the Dirichlet problem for the operator $AA^* + A_1^* A_1$ and the Dirichlet system $B_0^{(1)} = I$. In general situation we use the same method with minor modifications. Set $\tilde{m} = max(m, m_1)$ where m_1 is the order of A_1. Then the ellipticity of the complex (1.2) implies the existence of operators $Q_E \in \mathrm{Diff}_{\tilde{m}-m}(X; F \to B_E)$, $Q_F \in \mathrm{Diff}_{\tilde{m}-m_1}(X; F \to B_F)$ such that the operator $\mathfrak{A}_1 = Q_E A^* \oplus Q_G A_1$ has injective symbol and belong to $\mathrm{Diff}_{\tilde{m}}(X; F \to (B_E, B_G))$ (see [21, §6.4]). It is easy to guarantee that $S_{\mathfrak{A}_1}(\Omega) = S_{A^* \oplus A_1}(\Omega)$ for any $\Omega \subset X$. For $m = m_1$ we set $\mathfrak{A}_1 = A^* \oplus A_1$. Let us construct a suitable Dirichlet system $t_1 = \{B_j^{(1)}\}_{j=0}^{\tilde{m}-1}$ in order to consider the Dirichlet Problem for $\mathfrak{A}_1^* \mathfrak{A}_1$. If $\tilde{m} = m$ and $k = l$ then we set $t_1(f) = n(f)$ for $f \in C^\infty(\overline{D}, F)$. Otherwise, as $n(\cdot)$ consists of operators with surjective symbols (see [22, Lemma 9.2.5]) and there are such differential operators $\tilde{C}_j \in \mathrm{Diff}_{m-j-1}(U; F \to \tilde{F}_j^{(1)})$ $(0 \leq j \leq m-1)$ that symbols of the operators $C_j \oplus \tilde{C}_j$ are invertible on co-normal vectors to ∂D. Then for $0 \leq j \leq m-1$ we set $F_j^{(1)} = F_j \oplus \tilde{F}_j^{(1)}$ and $B_j^{(1)} f = C_{m-j-1} f \oplus \tilde{C}_{m-j-1} f$, for $f \in C^\infty(\overline{D}, F)$. The Dirichlet system $\{B_j^{(1)}\}_{j=0}^{m-1}$ of order $m-1$ may be completed easily to a Dirichlet system $t_1 = \{B_j^{(1)}\}_{j=0}^{\tilde{m}-1}$ of order $\tilde{m}-1$. Set $u_j^{(1)} = g_j \oplus 0 \in C^\infty(\partial D, F_j^{(1)})$, $u_j^{(1)} = 0$, $m \leq j \leq \tilde{m}-1$. It is clear now that $g = \mathcal{P}_1(\oplus_{j=0}^{m-1} u_j^{(1)})$, where $\mathcal{P}_1$ is the Poisson integral of the Dirichlet Problem for $\mathfrak{A}_1^* \mathfrak{A}_1$ and $t_1 = \{B_j^{(1)}\}_{j=0}^{m-1}$. Indeed, by the construction, $n(g) = \oplus_{j=0}^{m-1} g_j$; the estimate (1.5) and smoothness of the section g are guaranteed by Remark 1.3. □

Fix $0 \leq j \leq m-1$. By Lemma 1.5, for any $g_j \in C^\infty(\partial D, F_j)$ there is such a section $G_j \in C^\infty(\overline{D}, F)$ that $C_j G_j = g_j$, $C_i G_j = 0$ $(i \neq j)$ on ∂D and

$$\|G_j\|_{H^{m+s}(D,F)} \leq \gamma_j \|g_j\|_{H^{s+j+1/2}(\partial D, F_j)}.$$

Now Green formula (1.1) imply that for all $u \in C^\infty(\overline{D}, E)$ we have:

$$\int_{\partial D} (B_j u, g_j)_x ds = \int_D (Au, G_j)_x dx - \int_D (u, A^* G_j)_x dx,$$

$$\|u\|_{-s,t} \leq \left(1 + \|A^*_{m+s}\|^2 \sum_{j=0}^{m-1} \gamma_j^2 + \sum_{j=0}^{m-1} \gamma_j^2 \right)^{1/2} \|u\|_{-s,A},$$

which was to be proved. □

For the Laplacian A^*A and the Dirichlet system $\{t, n \circ A\}$ Theorem 1.4 was proved in [22, Theorem 9.3.6].

2 Weak boundary values of Sobolev functions

Consider now weak extension of the operator A on the scale $H^{-s}(D,E)$, $s \in \mathbb{Z}_+$. Denote $H^{-s}_{A,w}(D,E)$ the set of all sections $u \in H^{-s}(D,E)$ that admit $f \in H^{-s-m}(D,F)$ with $Au = f$ in $H(D,F,|\cdot|_{-s-m})$ (in particular, in D as a distribution). As A is linear operator, the set is linear too. Clearly

$$H^{-s}_A(D,E) \subset H^{-s}_{A,w}(D,E), \quad s \in \mathbb{Z}_+. \tag{2.1}$$

It is natural to expect $H^{-s}_{A,w}(D,E) = H^{-s}_A(D,E)$ (cf. [7]). We will prove it below. The union $\cup_{s=0}^{\infty} H^{-s}_{A,w}(D,E)$ will be denoted $H_A(D,E)$.

It is well-known (see, for instance, [6]), that for $s \geq m$, $s \in \mathbb{N}$ each Sobolev section $u \in H^s(D,E)$ has the trace $t(u) \in \oplus_{j=0}^{m-1} H^{s-j-1/2}(\partial D, F_j)$. For $s \leq m-1$ the situation is more subtle. It is known that any solution $u \in S_A(D) \cap H^s(D,E)$, $s \in \mathbb{Z}$ has on ∂D *weak limit* value $t(u) \in \oplus_{j=0}^{m-1} H^{s-j-1/2}(\partial D, F_j)$ (see [22, Lemma 9.4.4]). Theorem 1.4 implies $t_{-s}(u) \in \oplus_{j=0}^{m-1} H^{-s-j-1/2}(\partial D, F_j)$ for $u \in H^{-s}_A(D,E)$. Let us prove existence of traces for elements of $H^{-s}_{A,w}(D,E)$. With this aim, we define a pairing $(u,v)_D$ for $u \in H^{-s}(D,E)$ and $v \in C^\infty(\overline{D},E)$ in the following way. Take a sequence $\{u_\nu\}$ in $C^\infty(\overline{D},E)$ with $\|u_\nu - u\|_{-s} \to 0$ for $\nu \to \infty$. Then $|(u_\nu - u_\mu, v)_{L^2(D,E)}| \leq \|u_\nu - u_\mu\|_{-s} \|v\|_{H^s(D,E)} \to 0$ if $\mu, \nu \to \infty$. Set $(u,v)_D = \lim\limits_{\nu\to\infty} (u_\nu, v)_{L^2(D,E)}$. This limit does not depend on the choice of the sequence $\{u_\nu\}$, because if $\|u_\nu\|_{-s} \to 0$ for $\nu \to \infty$ then $|(u_\nu, v)_{L^2(D,E)}| \leq \|u_\nu\|_{-s} \|v\|_{H^s(D,E)}$ also tends to zero. By the definition,

$$|(u,v)_D| \leq \|u\|_{-s} \|v\|_{H^s(D,E)} \text{ for all } u \in H^{-s}(D,E), v \in C^\infty(\overline{D},E).$$

In a similar way we define the pairing $(u,v)_D$ for $u \in H(D,E,|\cdot|_{-s})$ and $v \in C^\infty_{m-1}(\overline{D},E)$; obviously, we have $|(u,v)_D| \leq |u|_{-s} \|v\|_{H^s(D,E)}$ too.

Let Γ be an open (in the topology of ∂D) connected subset of ∂D with piecewise smooth boundary $\partial\Gamma$ and let $H(D,F) = \cup_{s=0}^{\infty} H^{-s}(D,F)$.

Definition 2.1. Let $u \in H_A(D,E)$, $f \in H(D,F)$ satisfy $Au = f$ in D. We say that u has weak boundary value $t^w_\Gamma(u)$ of the Cauchy data with respect to A on Γ, coinciding with $\oplus_{j=0}^{m-1} u_j \in \oplus_{j=0}^{m-1} \mathcal{D}'(\Gamma, F_j)$ if

$$(f,g)_D - (u, A^* g)_D = \sum_{j=0}^{m-1} \langle \star C_j g, u_j \rangle_\Gamma, \text{ for all } g \in C^\infty_{comp}(D \cup \Gamma, F).$$

Green formula (1.1), and Theorem 1.4 imply that for any $u \in H^{-s}_A(D,E)$ there is weak boundary value $t^w_{\partial D}(u)$ coinciding with the trace $t_{-s}(u)$ from the space $\oplus_{j=0}^{m-1} H^{-s-j-1/2}(\partial D, F_j)$, $s \in \mathbb{Z}_+$. We want to connect weak boundary values with the so called *weak limit values*. More exactly, fix a defining function $\rho \in C^\infty$ for D and set $D_\varepsilon = \{x \in D : \rho(x) < -\varepsilon\}$. Without loss of the generality, we assume that $|\nabla \rho| = 1$ on ∂D. Then, for sufficiently small $\varepsilon > 0$, the sets $D_\varepsilon \Subset D \Subset D_{-\varepsilon}$ are domains with smooth boundaries $\partial D_{\pm\varepsilon} \in C^\infty$ and vectors $\mp \varepsilon \nu(x)$ belong to $\partial D_{\pm\varepsilon}$ for each point $x \in \partial D$ (here $\nu(x)$ is the exterior unit normal to the hyper surface ∂D at the point x). For a section $u \in C^m_{loc}(D,E)$ one says that $B_j u = u_j$ in the sense of weak limit values on Γ if

$$< u_j, v_j >_{\partial D} = \lim_{\varepsilon \to +0} \int_{\partial D} \langle v_j, B_j u(y - \varepsilon \nu(y)) \rangle_y ds_y \text{ for all } v_j \in C^\infty_{comp}(\Gamma, F^*_j).$$

Theorem 2.2. *Let both operators A and $A^* \oplus A_1$ have Property 1.1 in a neighborhood of $\overline{D}$. Then each section $u \in H^{-s}_{A,w}(D,E)$ has weak boundary value $t^w_{\partial D}(u)$ on Γ from the space $\oplus_{j=0}^{m-1} H^{-s-j-1/2}(\partial D, F_j)$ coinciding with weak limit value $\oplus_{j=0}^{m-1} B_j w$ of the section $w = (u - \mathcal{G}^{(D)} A^* f) \in S^F_{A^*A}(D)$. Moreover, this value does not depend on the choice of $f \in H^{-s-m}(D,F)$ with $Au = f$ in D.*

Proof. First of all we note that, according to Lemma 1.2, Theorem 1.4 and Remark 1.3, the operator $\mathcal{G}^{(D)}_{-s} A^*$ continuously maps $H^{-s-m}(D,F)$ to $H^{-s}_A(D,E)$. Hence every element w from the image $\mathcal{G}^{(D)}_{-s} A^*(H^{-s-m}(D,F))$ has zero trace $t_{-s}(w)$ and therefore zero boundary value on ∂D in the sense of Definition 2.1. It is clear, that the section $u \in H^{-s}_{A,w}(D,E)$ has weak boundary value $t^w_{\partial D}(u)$ if and only if the section $w = u - \mathcal{G}^{(D)}_{-s} A^* f$ has one. By the construction, $w \in H^{-s}_{A,w}(D,E)$ satisfies $A^*Aw = A^*Au - A^*f = 0$ in D. In particular, it is infinitely differentiable in D. Besides, as $w \in H^{-s}(D,E)$, this section has a finite order of growth near ∂D (see [19, Theorem 2.32]). Hence it has weak limit value $t(w) = \oplus_{j=0}^{m-1} u_j \in \oplus_{j=0}^{m-1} \mathcal{D}'(\partial D, F_j)$ (see [22, Theorem 9.4.8]).

As we noted above, $w \in H^{-s}_{A,w}(D,E)$ and $Aw = f - A\mathcal{G}^{(D)}_{-s} A^* f$ in D where $(f - A\mathcal{G}^{(D)}_{-s} A^* f) \in H^{-s-m}(D,F)$. In particular,

$$\langle \chi_D w, v\rangle = (w,v)_D \text{ for all } v \in C^\infty(\overset{\circ}{X}, E),$$

$$\langle \chi_D (f - A\mathcal{G}^{(D)}_{-s} A^* f), g\rangle = (f - A\mathcal{G}^{(D)}_{-s} A^* f, g)_D \text{ for all } g \in C^\infty(\overset{\circ}{X}, F).$$

Since $S_{\mathfrak{A}_1}(\Omega) = S_{A^*\oplus A_1}(\Omega)$ for $\Omega \subset \overset{\circ}{X}$ wee see that operator $\mathfrak{A}_1$ has Property 1.1. Thus, both sections w and Aw are solutions to elliptic operators with Property 1.1, i.e. $A^*Aw = 0$ in D, $\mathfrak{A}_1^*\mathfrak{A}_1(Aw) = 0$ in D. Moreover, they have finite orders of growth near ∂D. As Withney Theorem imply that any (infinitely) smooth section over $\overline{D}$ can be smoothly extended over X, it follows from [22, proof of Theorem 9.4.7] that there is such a sequence $\{\varepsilon_\nu\} \subset \mathbb{R}$ converging to zero that

$$(w,v)_D = \lim_{\varepsilon_\nu \to +0} \int_{D_{\varepsilon_\nu}} (w,v)_x dx \text{ for all } v \in C^\infty(\overline{D}, E),$$

$$(f - A\mathcal{G}^{(D)}_{s} A^* f, g)_D = \lim_{\varepsilon_\nu \to +0} \int_{D_{\varepsilon_\nu}} (Aw,g)_x dx \text{ for all } g \in C^\infty(\overline{D}, F).$$

But $t^w_{\partial D}(\mathcal{G}^{(D)}_{-s} A^* f) = 0$ on ∂D in the sense of Definition 2.1 and hence using Green formula we obtain for all $g \in C^\infty(\overline{D}, F)$ that

$$(f,g)_D - (u, A^*g)_D = (f - A\mathcal{G}^{(D)}_{-s} A^* f, g)_D - (w, A^*g)_D =$$

$$\lim_{\varepsilon_\nu \to +0} \left(\int_{D_{\varepsilon_\nu}} ((Aw,g)_x - (w,A^*g)_x) dx \right) = \sum_{j=0}^{m-1} \langle \star C_j g, u_j\rangle_{\partial D},$$

i.e. $t^w_{\partial D}(u) = t^w_{\partial D}(u - \mathcal{G}^{(D)}_{-s} A^* f)$ on ∂D.

Finally, if $\tilde f \in H(D,F)$ satisfies $Au = \tilde f$ in D then for $\tilde w = u - \mathcal{G}^{(D)} A^* \tilde f$ we have: $w - \tilde w = \mathcal{G}^{(D)} A^*(f - \tilde f) \in H_A(D,E)$ and $t^w_{\partial D}(w - \tilde w) = 0$ on ∂D, i.e. the trace $t^w_{\partial D}(u)$ does not depend on the choice of $f \in H(D,F)$ with $Au = f$ in D.

It remains to prove that the trace belong to the corresponding Sobolev space on ∂D. With this aim we fix some $0 \le j \le m-1$ and $g_j \in C^\infty(\partial D, F_j^*)$. Then, by Lemma 1.5, there is such a section $G_j \in C^\infty(\overline{D}, F)$ that $C_j G_j = g_j$, $C_i G_j = 0$ $(i \ne j)$ on ∂D and $\|G_j\|_{H^{m+s}(D,F)} \le \gamma_j \|g_j\|_{H^{s+j+1/2}(\partial D, F_j)}$. Hence, with the use of Definition 2.1, we obtain:

$$|\langle \star g_j, u_j\rangle_{\partial D}| = \left| \sum_{i=0}^{m-1} \langle \star C_i G_j u_i\rangle_{\partial D} \right| = |(f, G_j)_D - (u, A^* G_j)_D| \le$$

$$\|f\|_{-s-m}\|G_j\|_{H^{m+s}(D,F)} + \|u\|_{-s}\|A^*G_j\|_{H^s(D,E)}.$$

As for $s \geq 0$ the map A^*_{m+s} is continuous, the estimate (1.5) yields

$$|\langle u_j, g_j\rangle_{\partial D}| \leq \tilde{\gamma}_j(\|u\|_{-s} + \|f\|_{-s-m})\|g_j\|_{H^{s+j+1/2}(\partial D,F_j)}$$

with a positive constant $\tilde{\gamma}_j$ that does not depend on both g_j and u_j. Therefore

$$\|u_j\|_{H^{-s-j-1/2}(\partial D,F_j)} \leq \tilde{\gamma}_j(\|u\|_{-s} + \|f\|_{-s-m}).$$

This exactly means that $t^w_{\partial D}(u) \in \oplus_{j=0}^{m-1} H^{-s-j-1/2}(\partial D, F_j)$. □

Corollary 2.3. *The spaces $H^{-s}_A(D,E)$ and $H^{-s}_{A,w}(D,E)$, $s \in \mathbb{Z}_+$, coincide.*

Proof. Due to (2.1), it is sufficient to prove that $H^{-s}_{A,w}(D,E) \subset H^{-s}_A(D,E)$. Fix a section $u \in H^{-s}_{A,w}(D,E)$. Theorem 2.2 imply that u has weak boundary value $t^w_{\partial D}(u) \in \oplus_{j=0}^{m-1} H^{-s-j-1/2}(\partial D, F_j)$. We are going to show that u is a weak solution to Dirichlet problem (1.4) with data $F = A^*f \in H(D,E,|\cdot|_{-s-2m})$ and $t^w_{\partial D}(u) \in \oplus_{j=0}^{m-1} H^{-s-j-1/2}(\partial D, F_j)$. Indeed, according to Theorem 2.2,

$$(u, A^*Av)_D = (f, Av)_D - \sum_{j=0}^{m-1} \langle \star C_j Av, u_j\rangle_{\partial D} \text{ for all } v \in C^\infty_{m-1}(\overline{D}, E). \quad (2.2)$$

On the other hand, as $f \in H^{-s-m}(D,F)$, then there is a sequence $\{f_\nu\} \subset C^\infty(\overline{D},F)$ converging to f in this space. Using Lemma 1.2, we see that $\{A^*f_\nu\} \subset C^\infty(\overline{D},E)$ converges to A^*f in $H(D,E,|\cdot|_{-s-2m})$. That is why

$$(f, Av)_D = \lim_{\nu\to\infty}(f_\nu, Av)_D = \lim_{\nu\to\infty}(A^*f_\nu, v)_D = (A^*f, v)_D = (F, v)_D. \quad (2.3)$$

Taking into account (2.2) and (2.3) we conclude that u is the solution of the Dirichlet Problem. Finally, Remark 1.3 and [19, Theorem 2.30] imply that the section u belongs to $H^{-s}_t(D,E) = H^{-s}_A(D,E)$ (see Theorem 1.4). □

Thus, since weak and strong extensions of the operator A coincide then strong traces t_{-s} and weak boundary values $t^w_{\partial D}$ coincide too. Hence we will write simply t for trace operators on $H^{-s}_A(D,E)$, $s \geq 0$.

Corollary 2.4. *Let $u \in H_A(D,E)$. If $s \in \mathbb{Z}_+$, $Au \in H^{-s-m}(D,F)$, $t(u) \in \oplus_{j=0}^{m-1} H^{-s-j-1/2}(\partial D, F_j)$ then $u \in H^{-s}_A(D,E)$. Besides, if $Au \in H^{s-m}(D,F)$, $t(u) \in \oplus_{j=0}^{m-1} H^{s-j-1/2}(\partial D, F_j)$ then $u \in H^s(D,E)$.*

Proof. As we have seen proving Corollary 2.3, in this case u is a solution to Dirichlet problem (1.4) with data $F = A^*(Au) \in H(D, E, |\cdot|_{-s-2m})$ and $t(u) \in \oplus_{j=0}^{m-1} H^{-s-j-1/2}(\partial D, F_j)$. Now the statement of the lemma follows from [22, Theorem 9.3.17]). □

As we have seen above, a suitable class for solving boundary value problems for the operator A with data on the whole boundary ∂D is the space $H^{-s}_A(D, E)$, $s \in \mathbb{Z}_+$. In order to formulate the Cauchy problem for A we need a suitable space of the Cauchy data on $\Gamma \subset \partial D$. We choose *Sobolev spaces on closed subsets* of ∂D (see, for instance, [22, §1.1.3]). Namely, let $H^{-s}(\overline{\Gamma}, F_j)$ be a factor-space of the space $H^{-s}(\partial D, F_j)$ with respect to a subspace of sections vanishing in a neighbourhod of $\overline{\Gamma}$. By the very definition, each element of the space extends with the given Sobolev smoothness from Γ to ∂D. A further characterization of the spaces can be found, for instance, in [22, Lemma 12.3.2]). We only note that if $\partial\Gamma \in C^\infty$ then $H^{-s}(\Gamma, F_j) \hookrightarrow H^{-s}(\overline{\Gamma}, F_j) \hookrightarrow \tilde{H}^{-s}(\Gamma, F_j)$, $s \geq 0$.

Corollary 2.5. *For any $u \in H^{-s}_A(D, E)$, $s \in \mathbb{Z}_+$ and $\Gamma \subset \partial D$ there is boundary value $t^w_\Gamma(u)$ in the sense of Definition* 2.1 *belonging to $\oplus_{j=0}^{m-1} H^{-s-j-1/2}(\overline{\Gamma}, F_j)$.*

3 Green formula

Everywhere below we assume that the Laplacian A^*A has Property 1.1 on $\overset{\circ}{X}$. Then it has bilateral (i.e. left and right) pseudo differential fundamental solution, say Φ, on $\overset{\circ}{X}$ (see, for instance, [22, §4.4.2]). In particular, $\mathcal{L} = \Phi A^*$ is a left fundamental solution for A. Schwartz kernels of the operators Φ and $\mathcal{L}$ we denote $\Phi(x, y)$ and $\mathcal{L}(x, y)$ respectively, $x \neq y$. It is known that $\Phi(x, y) \in C^\infty((E \otimes E^*) \setminus \{x = y\})$ and $\mathcal{L}(x, y) = (A^*)'(y)\Phi(x, y)$ (see [21, §5]).

Now for $x \notin \partial D$ denote $M(\oplus_{j=0}^{m-1} v_j)(x)$ the Green transform with density $\oplus_{j=0}^{m-1} v_j \in \oplus_{j=0}^{m-1} \mathcal{D}'(\partial D, F_j)$, i.e. a result of action of the distribution $\oplus_{j=0}^{m-1} v_j$ on the test-function $(-\oplus_{j=0}^{m-1} C_j \mathcal{L}(x, \cdot)) \in \oplus_{j=0}^{m-1} C^\infty(\partial D, F_j^*)$. For a density $\oplus_{j=0}^{m-1} v_j \in \oplus_{j=0}^{m-1} C^\infty(\partial D, F_j)$ we have:

$$M(\oplus_{j=0}^{m-1} v_j)(x) = -\int_{\partial D} \sum_{j=0}^{m-1} \langle C_j(y)\mathcal{L}(x, y)), v_j\rangle_y \, ds_y, \qquad x \notin \partial D.$$

As the kernel $\mathcal{L}(x, y)$ is infinitely smooth with respect to x if $x \neq y$ then the Green transform is smooth everywhere on X outside the support supp $\oplus_{j=0}^{m-1} v_j$ of the density $\oplus_{j=0}^{m-1} v_j$. In particular, $M(\oplus_{j=0}^{m-1} v_j) \in S_{A^*A}(\overset{\circ}{X} \setminus \mathrm{supp}\,(\oplus_{j=0}^{m-1} v_j))$.

For a section $f \in C^\infty(\overline{D}, F)$ set $T_D f = \mathcal{L}\chi_D f$; this volume potential belongs to $S_{A^*A}(\overset{\circ}{X} \setminus \overline{D})$.

Lemma 3.1. *For any domain $\Omega \Subset \overset{\circ}{X}$ with $\partial\Omega \in C^\infty$ the potential T_D induces bounded linear operator $T_{D,\Omega} : H^{-s-m}(D, F) \to H^{-s}_A(\Omega, E)$, $s \in \mathbb{Z}_+$. Moreover, for any $f \in H^{-s-m}(D, F)$ the potential $T_{D,\Omega} f$ belongs to $S_{A^*A}(\Omega \setminus \overline{D})$.*

Proof. Any smoothing operator $\tilde{T}$ of the type $F \to E$ on $\overset{\circ}{X}$ induces a bounded linear operator $\tilde{T} : H^{-s-m}(D, F) \to C^p(\overline{\Omega}, E)$ for all p. As any two fundamental solutions on $\overset{\circ}{X}$ differ up to a smoothing operator, without loss of the generality we may assume that $\Phi = \mathcal{G}^{(X)}$. The main advantages of the use of $\mathcal{G}^{(X)}$ is that it is $L^2(X)$-self-adjoint (see, for instance, [19, equation (2.75)]) and it has the transmission property (see [13, §2.2.2]). That is why $\mathcal{G}^{(X)}\chi_\Omega\phi$ belongs to $H^m(X, E) \cap C^\infty(\overline{\Omega}, E)$ for all $\phi \in C^\infty(\overline{\Omega}, E)$. Similarly, $\mathcal{G}^{(X)} A^* \chi_\Omega \psi$ belongs to $H^m(X, E) \cap C^\infty(\overline{\Omega}, E)$ for all $\psi \in C^\infty(\overline{\Omega}, F)$. Then, for all $f \in C^\infty(\overline{D}, F)$, $v \in C^\infty(\overline{\Omega}, E)$, $g \in C^\infty(\overline{\Omega}, F)$ we have:

$$(T_D f, v)_\Omega = (\mathcal{G}^{(X)} A^* \chi_D f, \chi_\Omega v)_{\overset{\circ}{X}} = (\chi_D f, A\mathcal{G}^{(X)} \chi_\Omega v)_{\overset{\circ}{X}},$$

$$(AT_D f, g)_\Omega = (A\mathcal{G}^{(X)} A^* \chi_D f, \chi_\Omega g)_{\overset{\circ}{X}} = (\chi_D f, A\mathcal{G}^{(X)} A^* \chi_\Omega g)_{\overset{\circ}{X}}.$$

As the operator $\mathcal{G}^{(X)}$ is bounded on the scale of Sobolev spaces, we have:

$$\|T_D f\|_{-s,A,\Omega} \le C \, \|f\|_{-s-m,D} \text{ for all } f \in C^\infty(\overline{D}, F), \tag{3.1}$$

with constant $C > 0$ which does not depend on f.

Let now $f \in H^{-s-m}(D, F)$. Then there is a sequence $\{f_\nu\} \subset C^\infty(\overline{D}, F)$ approximating f in $H^{-s-m}(D, F)$. According to estimate (3.1), the sequence $\{T_D f_\nu\}$ is fundamental in $H^{-s}_A(\Omega, E)$; its limit we denote $T_{D,\Omega} f$. Clearly, this limit does not depend on the choice of the sequence $\{f_\nu\} \subset C^\infty(\overline{D}, F)$. Estimate (3.1) imply the boundedness of the defined linear operator $T_{D,\Omega}$. Besides, as each potential $T_D f_\nu$ belongs to $S_{A^*A}(\overset{\circ}{X} \setminus \overline{D})$, Stiltjes-Vitali Theorem implies that the sequence $\{T_D f_\nu\}$ converges uniformly with all the derivatives on compact subsets in $\Omega \setminus \overline{D}$ and its limit belongs to $S_{A^*A}(\Omega \setminus \overline{D})$. □

Lemma 3.2. *For any domain $\Omega \Subset \overset{\circ}{X}$ such that $\partial\Omega \in C^\infty$ and $D \subset \Omega$, the defined above potential M induces bounded linear operators*

$$M_D : \oplus_{j=0}^{m-1} H^{-s-j-1/2}(\partial D, F_j) \to H^{-s}_A(D, E),$$

$$M_\Omega : \oplus_{j=0}^{m-1} H^{-s-j-1/2}(\partial D, F_j) \to H^{-s}(\Omega, E), \quad s \in \mathbb{Z}_+.$$

Besides, for each $u \in H_A(D,E)$ Green formulas hold true:

$$M_D(t(u)) + T_{D,D}Au = u, \tag{3.2}$$

$$M_\Omega(t(u)) + T_{D,\Omega}Au = \chi_D u. \tag{3.3}$$

Proof. As we have noted above (see Remark 1.3), for any data $\oplus_{j=0}^{m-1} u_j$ from the space $\oplus_{j=0}^{m-1} H^{-s-j-1/2}(\partial D, F_j)$ the Poisson integral $\mathcal{P}(\oplus_{j=0}^{m-1} u_j) \in H_A^{-s}(D,E)$ satisfies $t(\mathcal{P}(\oplus_{j=0}^{m-1} u_j)) = \oplus_{j=0}^{m-1} u_j$. Set

$$M_D = \mathcal{P}^{(D)} - T_{D,D}A\mathcal{P}^{(D)} : \oplus_{j=0}^{m-1} H^{-s-j-1/2}(\partial D, F_j) \to H_A^{-s}(D,E),$$

$$M_\Omega = \chi_D \mathcal{P}^{(D)} - T_{D,\Omega}A\mathcal{P}^{(D)} : \oplus_{j=0}^{m-1} H^{-s-j-1/2}(\partial D, F_j) \to H^{-s}(\Omega,E).$$

Then Lemma 3.1, continuity of operators $\mathcal{P}^{(D)}$ and χ_D imply that the defined above linear operators M_D, M_Ω are continuous. Let us show that M_D and M_Ω coincide with M on $\oplus_{j=0}^{m-1} C^\infty(\partial D, F_j)$. Indeed, if $\oplus_{j=0}^{m-1} u_j \in \oplus_{j=0}^{m-1} C^\infty(\partial D, F_j)$ then it follows from Remark 1.3 that $\mathcal{P}^{(D)}(\oplus_{j=0}^{m-1} u_j) \in C^\infty(\overline{D}, E)$ and, in addition, $M(\oplus_{j=0}^{m-1} u_j) = M(t(\mathcal{P}^{(D)}(\oplus_{j=0}^{m-1} u_j)))$. Hence (the second) Green formula (see, for instance, [22, Lemma 10.2.3]) implies:

$$\chi_D \mathcal{P}^{(D)}(\oplus_{j=0}^{m-1} u_j) = M(\oplus_{j=0}^{m-1} u_j) + T_D A\mathcal{P}^{(D)}(\oplus_{j=0}^{m-1} u_j).$$

Since $\oplus_{j=0}^{m-1} C^\infty(\partial D, F_j)$ is dense in $\oplus_{j=0}^{m-1} H^{-s-j-1/2}(\partial D, F_j)$ then M can be extended from $\oplus_{j=0}^{m-1} C^\infty(\partial D, F_j)$ onto $\oplus_{j=0}^{m-1} H^{-s-j-1/2}(\partial D, F_j)$ by the continuity as maps to $H_A^{-s}(D,E)$ and $H^{-s}(\Omega,E)$ respectively. It is easy to see that the corresponding operators agree with M_D, M_Ω and with distribution $M(\oplus_{j=0}^{m-1} u_j) \in S_{A^*A}(\Omega \setminus \mathrm{supp}\,(\oplus_{j=0}^{m-1} u_j))$.

Now let $u \in H_A(D,E)$. Then $u \in H_A^{-s}(D,E)$ with a number $s \in \mathbb{Z}_+$ and there is a sequence $\{u_\nu\} \subset C^\infty(\overline{D},E)$ converging to u in $H_A^{-s}(D,E)$. Again the second Green formula imply

$$M(t(u_\nu)) + T_D Au_\nu = \chi_D u_\nu. \tag{3.4}$$

Passing to the limit with respect to $\nu \to \infty$ in (3.4) in the spaces $H_A^{-s}(D,E)$, $H^{-s}(\Omega,E)$ and using Lemma 3.1 and the already proved continuity of the operators M_D, M_Ω, we obtain formulas (3.2) and (3.3) respectively. □

Remark 3.3. Let $f \in H^{-s-m}(D,F)$, $s \in \mathbb{Z}_+$. If Ω, Ω_1 are bounded domains in $\mathring{X}$ containing D then $T_{D,\Omega} f \in S_{A^*A}(\Omega \setminus \overline{D})$ and $T_{D,\Omega_1} f \in S_{A^*A}(\Omega_1 \setminus \overline{D})$. Since each of them may be constructed as the limit of the same sequence, they coincide on $(\Omega_1 \cap \Omega) \setminus \overline{D}$. One can say the same on the potentials M_Ω M_{Ω_1}, because they were constructed with the use of $T_{D,\Omega}$, T_{D,Ω_1} respectively. Due to Property 1.1, this allows us to say on sections $T_D f \in S_{A^*A}(\overset{\circ}{X} \setminus \overline{D})$ and $M(\oplus_{j=1}^{m-1} u_j) \in S_{A^*A}(\overset{\circ}{X} \setminus \overline{D})$, having finite orders of growth near ∂D (outside $\overline{D}$) and such that $T_D f = T_{D,\Omega} f \in H^{-s}(\Omega, E)$, $M(\oplus_{j=1}^{m-1} u_j) = M_\Omega(\oplus_{j=1}^{m-1} u_j) \in H^{-s}(\Omega, E)$ for any domain $\Omega \supset D$.

4 The Cauchy problem

Set $\mathcal{D}'(\overline{\Gamma}, F_j) = \cup_{s=0}^{\infty} H^{-s-1/2}(\overline{\Gamma}, F_j)$ and consider the Cauchy problem.

Problem 4.1. *Given $f \in H(D, F)$, $\oplus_{j=0}^{m-1} u_j \in \oplus_{j=0}^{m-1} \mathcal{D}'(\overline{\Gamma}, F_j)$, find a section $u \in H_A(D, E)$ with $Au = f$ in D and $t(u) = \oplus_{j=0}^{m-1} u_j$ on Γ in the sense that*

$$(u, A^* g)_D = (f, g)_D - \sum_{j=0}^{m-1} \langle \star C_j g, u_j \rangle_\Gamma \textit{ for all } g \in C^\infty_{comp}(D \cup \Gamma, F). \quad (4.1)$$

Actually, Corollary 2.4 means that for sufficiently smooth data f and $\oplus_{j=0}^{m-1} u_j$ Problem 4.1 is the classical Cauchy problem for the operator A. Moreover, we easily obtain the Uniqueness Theorem for the problem.

Theorem 4.2. *If A has Property* 1.1 *in a neighborhood of $\overline{D}$ then Problem* 4.1 *can not have more than one solution.*

Proof. Indeed, let $u_0 = 0$, $f = 0$. Using Theorem 2.2 we see that a solution to Problem 4.1 belongs to $S_A(D)$ and it satisfies $\oplus_{j=0}^{m-1} B_j u_{|\Gamma} = 0$ in the sense of limit boundary values. As it has a finite order of growth near Γ (see [18, Theorem 2.6]), then $u \equiv 0$ in D according to [18, Theorem 2.8]). □

Now we note that properties of the complex (1.2) imply $A_1 f = 0$ in D if the Cauchy problem is solvable. Besides, each overdetermined operator A induces a tangential operator A_τ on ∂D (see [21, §11]). This means that the Cauchy data $\oplus_{j=0}^{m-1} u_j$ and f should be coherent in a sense. Namely, taking $g = A_1^* w$ with $w \in C^\infty_{comp}(D \cup \Gamma, G)$ in (4.1) and using the identity $A^* A_1^* \equiv 0$ we see that solvability of Problem 4.1 implies

$$\sum_{j=0}^{m-1} \langle \star C_j A_1^* w, u_j \rangle_\Gamma = (f, A_1^* w)_D \text{ for all } w \in C^\infty_{comp}(D \cup \Gamma, G). \quad (4.2)$$

If A is a multi-dimensional Cauchy-Riemann operator and $f = 0$ then condition (4.2) is the well-known tangential Cauchy-Riemann condition (see [3]).

Let us obtain a solvability criterion for Problem 4.1. With this aim, we choose a domain D^+ in such a way that the set $\Omega = D \cup \Gamma \cup D^+$ is a domain with infinitely smooth boundary. It is convenient to denote $D^- = D$. For a section ϕ over Ω, we denote $\phi^\pm$ the restriction of ϕ on $D^\pm$. Now for the boundary data $\oplus_{j=0}^{m-1} u_j \in \oplus_{j=0}^{m-1} H^{-s-j-1/2}(\overline{\Gamma}, F_j)$ fix a representative $\oplus_{j=0}^{m-1} \tilde{u}_j \in \oplus_{j=0}^{m-1} H^{-s-j-1/2}(\partial D, F_j)$, $s \geq 0$. Clearly, the potentials $M(\oplus_{j=0}^{m-1} \tilde{u}_j)$ and $T_D f$ belong to $S_{A^*A}(D^+)$ as parameter depending distributions. Hence the section

$$F = M(\oplus_{j=0}^{m-1} u_j) + T_D f$$

belongs to $S_{A^*A}(D^+)$ (see Remark 3.3). Taking into account Green formula (3.3) we see that F may contain a lot of information on the solution of Problem 4.1, if it exists.

Theorem 4.3. *Let both A^*A and $A_1 \oplus A^*$ have Property* 1.1. *The Cauchy Problem* 4.1 *is solvable if and only if condition* (4.2) *holds and there is a section $\mathcal{F} \in S^F_{A^*A}(\Omega)$ coinciding with F in D^+.*

Proof. Let Problem 4.1 be solvable and u be its solution. The necessity of (4.2) was already shown above. Set

$$\mathcal{F} = M_\Omega(\oplus_{j=0}^{m-1} \tilde{u}_j) + T_{D,\Omega} f - \chi_D u. \tag{4.3}$$

Now Lemmata 3.1, 3.2 and Remark 3.3 imply that $\mathcal{F} \in H^{-s}(\Omega, E)$ with some $s \in \mathbb{Z}_+$. It satisfies $A^*A\mathcal{F} = 0$ in D^+ and coincides with F^+ there. Using Green formula (3.3) and Lemma 3.2 we obtain $\mathcal{F} = M_\Omega(\oplus_{j=0}^{m-1}(\tilde{u}_j - B_j u))$. Since $\oplus_{j=0}^{m-1}(\tilde{u}_j - B_j u) = 0$ on Γ then $M_\Omega(\oplus_{j=0}^{m-1}(\tilde{u}_j - B_j u))$ belongs to $S_{A^*A}(\overset{\circ}{X} \setminus \Gamma) \subset S_{A^*A}(\Omega)$ as a parameter depending distribution. Hence $\mathcal{F}$ has the same property. Moreover, Seeley's Theorem on the kernel (see [21, Theorem 5.10]) imply that $\Phi(x, y)$ is infinitely differentiable outside the diagonal $\{x = y\}$ and it has on the diagonal the same order of the singularity as $|x - y|^{2m-n}$. Then the compactness of ∂D guarantees finite order of growth for $M_\Omega(\oplus_{j=0}^{m-1}(\tilde{u}_j - B_j u))$.

Back, let there be $\mathcal{F} \in S^F_{A^*A}(\Omega)$ coinciding with F in D^+. Set

$$u = T_{D,D} f + M_D(\oplus_{j=0}^{m-1} \tilde{u}_j) - \mathcal{F}^-. \tag{4.4}$$

As $f \in H(D, F)$ and $\oplus_{j=0}^{m-1} u_j \in \oplus_{j=0}^{m-1} \mathcal{D}'(\overline{\Gamma}, F_j)$ then Lemmata 3.1, 3.2 imply that $T_{D,D} f + M_D(\oplus_{j=0}^{m-1} \tilde{u}_j) \in H_A(D, E)$. Moreover, since $D \subset \Omega$, we see that $\mathcal{F}^- \in S^F_{A^*A}(D)$, i.e. $t(\mathcal{F}^-) \in \oplus_{j=0}^{m-1}\mathcal{D}'(\partial D, F_j)$ (see [19, Theorem 2.32]). Therefore $\mathcal{P}^{(D)}(t(\mathcal{F}^-)) = \mathcal{F}^- \in H_A(D,E)$ (see Remark 1.3). Thus, by the very construction, the section u belongs to $H_A(D,E)$. According to Theorem 2.2, there is trace $t(u)$ on $\overline{\Gamma}$ in the space of distributions; it can be found with the use of Definition 2.1. Indeed, since $f \in H(D,F)$ there is such $p \in \mathbb{Z}_+$ that $f \in H^{-p}(D,F)$. Let $\{f_\nu\} \subset C^\infty(\overline{D},F)$, $\{u_j^{(\nu)}\} \subset C^\infty(\partial D, F_j)$ be sequences approximating distributions $f \in H^{-p}(D,F)$ and $u_j \in \mathcal{D}'(\partial D, F_j)$ in the corresponding spaces respectively. Then (see [21, Theorem 13.5])

$$B_j(T_D f_\nu)^- - B_j(T_D f_\nu)^+ = 0 \text{ on } \Gamma, \qquad 0 \le j \le m-1,$$

$$B_j(M^-(\oplus_{i=0}^{m-1}\tilde{u}_i^{(\nu)}) - M^+(\oplus_{i=0}^{m-1}\tilde{u}_i^{(\nu)})) = u_j^{(\nu)} \text{ on } \Gamma, \qquad 0 \le j \le m-1.$$

Now Green formula (1.1), Lemmata 1.2, 3.1, 3.2 and Remark 3.3 imply that $t(u) = \oplus_{j=0}^{m-1} u_j$ on Γ, because for any section $g \in C^\infty_{comp}(D\cup\Gamma, F)$ we have:

$$(Au,g)_D - (u,A^*g)_D =$$

$$\lim_{\nu\to\infty}\sum_{j=0}^{m-1}(B_j((T_D f_\nu)^- + M^-(\oplus_{i=0}^{m-1}\tilde{u}_i^{(\nu)}) - \mathcal{F}), C_j g)_{\partial D} =$$

$$\lim_{\nu\to\infty}\sum_{j=0}^{m-1}(\tilde{u}_j^{(\nu)}, C_j g)_{\partial D} = \sum_{j=0}^{m-1}\langle \star C_j g, u_j\rangle_\Gamma.$$

In order to finish the proof we need to check that $Au = f$ in D. With this aim, consider section $h = \chi_D(f - Au)$ belonging to $H(\Omega, F)$. Clearly, $C^\infty_{comp}(\Omega, G) \subset C^\infty_{comp}(D\cup\Gamma, G)$ and hence, by condition (4.2), the identity $A_1 \circ A \equiv 0$ and Definition 2.1 we have for all $w \in C^\infty_{comp}(\Omega, G)$:

$$(h, A_1^* w)_\Omega = (f, A_1^* w)_D - (Au, A_1^* w)_D =$$

$$\sum_{j=0}^{m-1}\langle \star C_j A_1^* w, u_j\rangle_{\partial D} - \sum_{j=0}^{m-1}\langle \star C_j A_1^* w, u_j\rangle_{\partial D} + (u, A^* A_1^* w)_D = 0.$$

Thus, $A_1 h = 0$ in Ω. On the other hand, according to Definition 2.1, for all $v \in C^\infty_{comp}(\Omega, E)$, we have:

$$(h, Av)_\Omega = (f, Av)_D - (u, A^*Av)_D - \sum_{j=0}^{m-1}\langle \star C_j Av, u_j\rangle. \tag{4.5}$$

As $\mathcal{F} \in S^F_\Delta(\Omega)$ equals to F in D^+, Remark 3.3 imply, for all $v \in C^\infty_{comp}(\Omega, E)$,

$$(\mathcal{F}, A^*Av)_D = -(\mathcal{F}, A^*Av)_{D^+} = -(T_{D,\Omega}f + M_\Omega(\oplus_{j=0}^{m-1}\tilde{u}_j), A^*Av)_{D^+}. \quad (4.6)$$

Further, as Φ is a bilateral fundamental solution for A^*A then for $f \in C^\infty(\overline{D}, F)$ the identity $A^*AT_D f = A^*\chi_D f$ holds true. Hence, taking a sequence $\{f_\nu\} \subset C^\infty(\overline{D}, F)$ approximating $f \in H^{-p}(D, F)$ in this space and using Lemmata 1.2 and 3.1 we see that, for all $v \in C^\infty_{comp}(\Omega, E)$, we have:

$$(T_{D,\Omega}f, A^*Av)_{D^+} + (T_{D,D}f, A^*Av)_D =$$

$$\lim_{\nu\to\infty}(T_D f_\nu, A^*Av)_\Omega = \lim_{\nu\to\infty}(A^*\chi_D f_\nu, v)_\Omega = (f, Av)_D. \quad (4.7)$$

Take a domain $\tilde{\Omega} \supset \Omega$ with $D \Subset \tilde{\Omega}$. Then it follows from Lemma 3.2 that $M(\oplus_{j=0}^{m-1}\tilde{u}_j)$ belongs to $H_{A^*A}(\tilde{\Omega} \setminus \overline{D}, E)$ and $H_{A^*A}(D, E)$. Using (4.5), (4.6), (4.7), we conclude that, for all $v \in C^\infty_{comp}(\Omega, E)$,

$$(h, Av)_\Omega = (M(\oplus_{j=0}^{m-1}\tilde{u}_j), \Delta v)_{D^+} + (M(\oplus_{j=0}^{m-1}\tilde{u}_j), \Delta v)_D - \sum_{j=0}^{m-1}\langle\star C_j Av, u_j\rangle. \quad (4.8)$$

Now choose as a Dirichlet system of order $2m-1$ the system $\{B_j, C_j A\}$; its adjoint system with respect to Green formula is again $\{B_j, C_j A\}$ (see [18, Lemma 9.2]). As A^*A is elliptic and formally self-adjoint, the corresponding compatibility operator equals to zero and hypotheses of Theorem 2.2 are fulfilled. Now, using Definition 2.1 and Theorem 2.2 for A^*A, we see that for all $v \in C^\infty_{comp}(\Omega, E)$ we have:

$$(M^+(\oplus_{j=0}^{m-1}\tilde{u}_j), A^*Av)_{D^+} + (M^-(\oplus_{j=0}^{m-1}\tilde{u}_j), A^*Av)_D =$$

$$\sum_{j=0}^{m-1}(\tilde{u}_j, C_j Av)_{\partial D} = \sum_{j=0}^{m-1}\langle\star C_j Av, u_j\rangle_\Gamma, \quad (4.9)$$

because according to [18, Lemma 2.7], in the sense of weak limit values on Γ, we have:

$$t(M^-(\oplus_{j=0}^{m-1}\tilde{u}_j) - (M^+(\oplus_{j=0}^{m-1}\tilde{u}_j) = \oplus_{j=0}^{m-1}u_j,$$

$$n(A(M^-(\oplus_{j=0}^{m-1}\tilde{u}_j) - M^+(\oplus_{j=0}^{m-1}\tilde{u}_j))) = 0.$$

Thus, (4.8), (4.9) yield $A^*h = 0$ in Ω and hence $(A_1 \oplus A^*)h = 0$ in Ω. Finally, Property 1.1 for $A_1 \oplus A^*$ means that $h \equiv 0$ in Ω. In particular, $f = Au$ in D. $\square$

Corollary 4.4. *Let both operators A^*A and $A_1 \oplus A^*$ have Property* 1.1. *If $f \in H^{-s-m}(D,F)$, $\oplus_{j=0}^{m-1} u_j \in \oplus_{j=0}^{m-1} H^{-s-j-1/2}(\overline{\Gamma}, F_j)$, $s \in \mathbb{Z}_+$ then the Cauchy problem* 4.1 *is solvable in $H_A^{-s}(D,E)$ if and only if condition* (4.2) *holds and there is a section $\mathcal{F} \in S_{A^*A}(\Omega) \cap H^{-s}(\Omega,E)$ coinciding with F in D^+.*

Proof. Indeed, if Problem 4.1 is solvable in $H_A^{-s}(D,E)$ then it is solvable in $H_A(D,E)$. Hence $\mathcal{F} = M_\Omega(\oplus_{j=0}^{m-1}\tilde{u}_j - t(u))$ (see the proof of Theorem 4.3). Therefore, using Lemma 3.2 we see that $\mathcal{F}$ belongs to $H^{-s}(\Omega,E)$.

Back, if $\mathcal{F} \in H^{-s}(\Omega,E) \cap S_{A^*A}(\Omega)$ coincides with F in D^+ then Problem 4.1 is solvable. Its unique solution u is given by formula (4.4) and $\mathcal{F}$ is given by formula (4.3). In particular, $\chi_D u = (F - \mathcal{F})$ belongs to $H^{-s}(\Omega,E)$. Let us take $v \in C^\infty(\overline{D},E)$. Then there is a section $V \in C^\infty(\overline{\Omega},E)$ with $\|V\|_{s,\Omega} = \|v\|_{s,D}$ and $v = V$ in D. By the definition,

$$|(u,v)_D| = |(\chi_D u, V)_\Omega| \leq \|\chi_D u\|_{-s,\Omega}\|v\|_{s,D},$$

i.e. $u \in H^{-s}(D,E)$. As $Au = f \in H^{-s-m}(D,F)$, it yields $u \in H_A^{-s}(D,E)$. □

For $s \geq 0$, $f = 0$ and an operator A included to an elliptic complex (1.2), being exact on the level of sheaves, this corollary was proved in [18].

5 Carleman formula

We will use the so called bases with the double orthogonality property (see [12], [16]) in order to construct Carleman formula for Problem 4.1. (cf. [3], [18]). Denote $h^s(\Omega)$ the space $S_{A^*A}(\Omega) \cap H^s(\Omega,E)$ with $s \in \mathbb{Z}$.

Lemma 5.1. *Let $p, s \in \mathbb{Z}$. If $\omega \Subset \Omega$ is a domain with smooth boundary and $\Omega \setminus \omega$ has no compact components then there is orthonormal basis $\{b_\nu\}_{\nu=1}^\infty$ in $h^s(\Omega)$ such that $\{b_{\nu|\omega}\}_{\nu=1}^\infty$ is orthogonal basis in $h^p(\omega)$.*

Proof. In fact, the sections $\{b_\nu\}_{\nu=1}^\infty$ are eigen-vectors of compact self-adjoint operator $R(\Omega,\omega)^*R(\Omega,\omega)$ with $R(\Omega,\omega) : h^s(\Omega) \to h^p(\omega)$ being the natural embedding operator (see, for instance, [16], [12], [18, Theorem 6.5]). We need to prove the compactness of the operator $R(\Omega,\omega)$ only; however the arguments are the same as in [18, Lemma 6.4]). □

Let us use the basis $\{b_\nu\}_{\nu=1}^\infty$ in order to simplify Corollary 4.4. We fix s, p, and domains $\omega \Subset D^+$, Ω as in Lemma 5.1 and set $c_\nu(\phi) = \frac{(\phi,b_\nu)_p}{\|b_\nu\|_p^2}$, $\nu \in \mathbb{N}$.

Corollary 5.2 (See [18], Example 1.9). *Let $\phi \in h^p(\omega)$. Then there is $\Phi \in h^s(\Omega)$ coinciding with ϕ in ω if and only if the series $\sum\limits_{\nu=1}^{\infty} |c_\nu(\phi)|^2$ converges.*

Corollary 5.3. *Let $s \in \mathbb{Z}_+$, $p \in \mathbb{Z}$ and both A^*A and $A^* \oplus A_1$ have Property* 1.1. *The Cauchy Problem* 4.1 *is solvable in $H_A^{-s}(D,E)$ if and only if condition* (4.2) *holds and the series $\sum\limits_{\nu=1}^{\infty} |c_\nu(F^+)|^2$ converges.*

Proof. Follows from Corollaries 4.4 and 5.2. In particular, we have

$$\mathcal{F}(x) = \sum_{\nu=1}^{\infty} c_\nu(F^+) b_\nu(x), \qquad x \in \Omega \tag{5.1}$$

in the space $h^s(\Omega)$. □

Let us obtain Carleman formula for solutions to Problem 4.1. With this aim, consider Carleman kernel:

$$\mathfrak{C}_N(y,x) = \mathcal{L}(y,x) - \sum_{\nu=1}^{N} c_\nu(\mathcal{L}(y,\cdot)) b_\nu(x),\ N \in \mathbb{N},\ x \in \Omega,\ y \notin \overline{\omega}, x \neq y.$$

Corollary 5.4. *Let both A^*A and $A^* \oplus A_1$ have Property* 1.1. *Then for any $u \in H_A^{-s}(D,E)$, $s \geq 0$, the Carleman formula holds:*

$$\lim_{N\to\infty} \left\| u - \sum_{j=0}^{m-1} (\tilde{u}_j, C_j \star_F^{-1} \mathfrak{C}_N(.,x))_{\partial D} + (Au, \star_F^{-1} \mathfrak{C}_N(.,x))_D \right\|_{-s,A,D} = 0, \tag{5.2}$$

where $\tilde{u}_j$ is an extension of $B_j u$ from $\overline{\Gamma}$ onto ∂D belonging $H^{-s-j-1/2}(\partial D, F_j)$.

Proof. For the Cauchy data $f = Au$ and $\oplus_{j=0}^{m-1} u_j = (B_j u)_{|\Gamma}$ the Cauchy problem 4.1 is solvable in $H_A^{-s}(D,E)$. Hence Theorems 4.2 and 4.3 imply that its unique solution u is given by equation (4.4). As $\overline{\omega} \cap \overline{D} = \emptyset$, using Fubini Theorem we see that for all $\nu \in \mathbb{N}$:

$$c_\nu(F^+) = (f, \star_F^{-1} c_\nu(\mathcal{L}(y,.))_D - \sum_{j=0}^{m-1} (\tilde{u}_j, C_j(y) \star_F^{-1} (y) c_\nu(\mathcal{L}(y,.))_{\partial D}.$$

Finally, applying Corollary 5.3, equation (5.1) and regrouping summands in equation (4.4) we obtain the statement of the corollary. □

Examples of bases with the double orthogonality property could be found, for instance, in [16], [17], [18], [22]. Let us see one of them for the Laplace operator.

Example 5.5. Let $\{h_\nu^{(i)}\}$ be the set of homogeneous harmonic polynomials forming an orthonormal basis in $L^2(\partial B(0,1))$ on the unit sphere $\partial B(0,1)$ in $\mathbb{R}^n$, $n \geq 2$ (see [20, p. 453]), where ν is the degree of homogeneity and i is number of the polynomial of degree ν in the basis, $1 \leq i \leq J(\nu)$, $J(\nu) = \frac{(n+2\nu-2)(n+\nu-3)!}{\nu!(n-2)!}$, $\nu > 0$, $J(0) = 1$. It is easy to see that the system $\{h_\nu^{(i)}\}$ is orthogonal in $h^s(B(0,R))$, $s \in \mathbb{Z}_+$, for any ball $B(0,R)$. Let us show that for any $R > 0$ the system $\{h_\nu^{(i)}\}$ is orthogonal in the space $h^{-s}(B(0,R))$, $s \in \mathbb{Z}_+$, with the scalar product $(u,v)_{-s,B(0,R)}$. Consider a typical summand $\|h_\nu^{(i)} \pm h_\mu^{(j)}\|^2_{-s}$ in (1.3). Then, by direct calculation, using $L^2(B(0,R))$-orthogonality of the polynomials,

$$\|h_\nu^{(i)} \pm h_\mu^{(j)}\|^2_{-s} = \sup_{|c_\nu^{(i)}|^2\|h_\nu^{(i)}\|^2_s \pm |c_\mu^{(j)}|^2\|h_\mu^{(j)}\|^2_s = 1} |c_\nu^{(i)}\|h_\nu^{(i)}\|^2_{0,B(0,R)} \pm c_\mu^{(j)}\|h_\mu^{(j)}\|^2_{0,B(0,R)}|^2.$$

Then, because of symmetry reasons, $\|h_\nu^{(i)} + h_\mu^{(j)}\|^2_{-s} - \|h_\nu^{(i)} - h_\mu^{(j)}\|^2_{-s} = 0$.

Example 5.6. If A is the Laplace operator Δ_n in $\mathbb{R}^n$ then $A^* = \Delta_n$, $A^*A = \Delta_n^2$, $A_1 \equiv 0$, $A^* \oplus A_1 = \Delta_n$. Set $B_0 = 1$, $B_1 = \frac{\partial}{\partial \nu}$. Then the Cauchy problem 4.1 consists of recovering a function $u \in H_\Delta^{-s}(D)$, $s \in \mathbb{Z}_+$, via its Laplacian $\Delta u \in H^{-s-2}(D)$ and traces on Γ of $u \in H^{-s-1/2}(\overline{\Gamma})$ and its normal derivative $\frac{\partial u}{\partial \nu} \in H^{-s-3/2}(\overline{\Gamma})$. Property 1.1 for A^*A and $A^* \oplus A_1$ hold true because Uniqueness Theorems for harmonic and biharmonic functions. The compatibility condition (4.2) is trivial for all u_0, u_1, f. As a left fundamental solution for Δ one can take the standard one: $g_n(x)\frac{|x|^{2-n}}{(2-n)\sigma_n}$ where σ_n is the square of the unite sphere in $\mathbb{R}^n$, if $n > 2$, and $g_2(x) = \ln|x|$, if $n = 2$. Then (3.3) is the standard Green formula for the Laplace operator and the potentials $T_D f$, $M(u_0,u_1)$ are harmonic in $\mathbb{R}^n \setminus \overline{D}$. Thus the Cauchy problem 4.1 is equivalent to the harmonic extension of $T_D f + M(u_0,u_1)$ from D^+ to Ω. Let D be a part of the unit ball Ω, cut off by a hyper surface $\Gamma \not\ni 0$. Then example yields that formula (5.2) for the Laplace operator (cf. [17] for $\Delta v = 0$) can be extended to negative Sobolev spaces:

$$u^{(N)}(x) = \int_{\partial D} \left(\tilde{u}_0(y) \frac{\partial \mathfrak{C}_N^\Delta(y,x)}{\partial \nu_y} - \tilde{u}_1(y)\mathfrak{C}_N^\Delta(y,x) \right) ds_y + \int_D \mathfrak{C}_N^\Delta(y,x)\Delta v(y)dy,$$

$$\mathfrak{C}_N^\Delta(y,x) = g_n(x-y) - g_n(y) - \sum_{\mu=1}^{N} \sum_{i=1}^{J(\mu)} \frac{\overline{h_\mu^{(i)}(y)} h_\mu^{(i)}(x)}{|y|^{n+2\mu-2}(n+2\mu-2)}, \quad N \in \mathbb{N}.$$

Example 5.7. If $A = \sum_{j=1}^n A^{(j)} \frac{\partial}{\partial x_j}$ is a Dirac operator in $\mathbb{R}^n$, i.e. it is a $(l \times k)$-operator with constant coefficients such that $A^* A = -\Delta_n I_k$ (here I_k is the unit $(k \times k)$-matrix, $l \geq k$), then the famous Hilbert Theorem yields that the compatibility operators A_i have constant coefficients too. If operators A_i are homogeneous then complex (1.2) is elliptic (see [21, Proposition 2.3]). Let $B_0 = 1$ and D be a part of the unit ball Ω, cut off by a hyper surface $\Gamma \not\ni 0$. For $l > k$ condition (4.2) may be non-trivial. The examples 5.5 and 5.6 imply that formula (5.2) for Dirac operators (cf. [18] for $Av = 0$) can be extended to negative Sobolev spaces:

$$u^{(N)}(x) = -\int_{\partial D} \mathfrak{C}_N^A(y,x) \sum_{j=1}^n A^{(j)} \tilde{u}_0(y) dy[j] + \int_D \mathfrak{C}_N^A(y,x) Au(y) y, \quad N \in \mathbb{N},$$

$\mathfrak{C}_N^A(y,x) = \sum_{i=1}^n \frac{\partial}{\partial y_i} \mathfrak{C}_N^\Delta(y,x) A^{(i)^*}$, $dy[j] = dy_1 \wedge \ldots dy_{j-1} \wedge dy_{j+1} \wedge \ldots \wedge dy_n$.

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