On the Cauchy Problem for the First Order Elliptic Complexes in Spaces of Distributions

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Abstract

Let D be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with a smooth boundary ∂D. We indicate appropriate Sobolev spaces of negative smoothness to study the non-homogeneous Cauchy problem for an elliptic differential complex $\{A_i\}$ of first order operators. In particular, we describe traces on ∂D of tangential part $\tau_i(u)$ and normal part $\nu_i(u)$ of a (vector)-function u from the corresponding Sobolev space and give an adequate formulation of the problem. If the Laplacians of the complex satisfy the uniqueness condition in the small then we obtain necessary and sufficient solvability conditions of the problem and produce formulae for its exact and approximate solutions. For the Cauchy problem in the Lebesgue space $L^2(D)$ we construct the approximate and exact solutions to the Cauchy problem with the maximal possible regularity. Moreover, using Hilbert space methods, we construct Carleman's formulae for a (vector-) function u from the Sobolev space $H^1(D)$ by its Cauchy data $\tau_i(u)$ on a subset $\Gamma \subset \partial D$ and the values of $A_i u$ in D modulo the null-space of the Cauchy problem. Some instructive examples for elliptic complexes of operators with constant coefficients are considered.

Key words: Elliptic differential complexes, ill-posed Cauchy problem, Carleman's formula.

It is well known that the Cauchy problem for an elliptic system A is ill-posed (see, for instance, [1]). Apparently, the serious investigation of the problem was stimulated by practical needs. Namely, it naturally appears in applications: in hydrodynamics (as the Cauchy problem for holomorphic functions), in geophysics (as the Cauchy problem for the Laplace operator), in elasticity theory (as the Cauchy problem for the Lamé system) etc., see, for instance, the book [2] and its bibliography. The problem was actively studied through the XX century (see, for instance, [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13] and many others).

Differential complexes appear as compatibility conditions for overdetermined operators (see, for instance, [14], [15]). Thus, the Cauchy problem for them is of the special interest. One of the first problems of this kind was the Cauchy problem for the Dolbeault complex (the compatibility complex for the multidimensional Cauchy-Riemann system), see [16]. The interest to it was great because of the famous example by H. Lewy of the differential equation without solutions, constructed with the use of the tangential Cauchy-Riemann operator, see [17]. Recently new approaches to the problem were found in spaces of smooth functions (see [18], [19]) and in spaces of distributions (see [20], [21]).

We consider the Cauchy problem in spaces of distributions with some restrictions on growth in order to correctly define its traces on boundaries of domains (see, for instance, [2], [22], [23], [24], [25],). In this paper we develop the approach presented in [9] to study the *homogeneous* Cauchy problem for overdetermined elliptic partial differential operators. Instead we consider the *non-homogeneous* Cauchy problem for elliptic complexes.

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1 Preliminaries

1.1 Differential complexes

Let X be a C^∞-manifold of dimension $n \geq 2$ with a smooth boundary ∂X. We tacitly assume that it is enclosed into a smooth closed manifold $\tilde{X}$ of the same dimension.

For any smooth $\mathbb{C}$-vector bundles E and F over X, we write $\mathrm{Diff}_m(X; E \to F)$ for the space of all the linear partial differential operators of order $\leq m$ between sections of E and F. Then, for an open set $O \subset \overset{\circ}{X}$ (here $\overset{\circ}{X}$ is the interior of X) over which the bundles and the manifold are trivial, the sections over O may be interpreted as (vector-) functions and $A \in \mathrm{Diff}_m(X; E \to F)$ is given as $(l \times k)$-matrix of scalar differential operators, i.e. we have

$$A = \sum_{|\alpha| \leq m} a_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}, \quad x \in O,$$

where $a_\alpha(x)$ are $(l \times k)$-matrices of $C^\infty(O)$-functions, $k = \mathrm{rank}(E)$, $l = \mathrm{rank}(F)$.

Denote E^* the conjugate bundle of E. Any Hermitian metric $(.,.)_x$ on E gives rise to a sesquilinear bundle isomorphism (the Hodge operator) $\star_E \colon E \to E^*$ by the equality $\langle \star_E v, u \rangle_x = (u, v)_x$ for all sections u and v of E; here $\langle ., . \rangle_x$ is the natural pairing in the fibers of E^* and E.

Pick a volume form dx on X, thus identifying the dual and the conjugate bundles. For $A \in \mathrm{Diff}_m(X; E \to F)$, denote by $A^* \in \mathrm{Diff}_m(X; F \to E)$ the formal adjoint operator.

Let $\pi : T^*X \to X$ be the (real) cotangent bundle of X and let π^*E be a induced bundle for the bundle E (i.e. the fiber of π^*E over the point $(x, z) \in T^*X$ coincides with E_x). We write $\sigma(A) : \pi^*E \to \pi^*F$ for the principal homogeneous symbol of the order m of the operator A.

Let D be a bounded domain (i.e. open connected set) in $\overset{\circ}{X}$ with infinitely differentiable boundary ∂D. Denote $C^\infty(D, E)$ the Fréchet space of all the infinitely differentiable sections of the bundle E over D and denote $C^\infty(\overline{D}, E)$ the subset in $C^\infty(D, E)$ which consists of sections with all the derivatives continuously extending up to $\overline{D}$. Let also $C^\infty_{comp}(D, E)$ stand for the set of all the smooth sections with compact supports in D. Besides, for an open (in the topology of ∂D) subset $\Gamma \subset \partial D$, let $C^\infty_{comp}(D \cup \Gamma, E)$ be the set of all the $C^\infty(\overline{D}, E)$-sections with compact supports in $D \cup \Gamma$.

For a distribution-section $u \in (C^\infty_{comp}(D, E))'$ we always understand Au in the sense of distributions in D. The spaces of all the weak solutions of the operator A in D we denote $S_A(D)$.

We often refer to the so-called uniqueness condition in the small on $\overset{\circ}{X}$ for an operator A.

Condition 1.1. *If u is a distribution in a domain $D \Subset \overset{\circ}{X}$ with $Au = 0$ in D and $u = 0$ on an open subset O of D then $u \equiv 0$ in D.*

It holds true if, for instance, all the objects under consideration are real analytic.

Let $G_A(.,.) \in \mathrm{Diff}_{m-1}(X; (F^*, E) \to \Lambda^{n-1})$ denote a Green operator attached to A, i.e. such a bi-differential operator that

$$dG_A(\star_F g, v) = ((Av, g)_x - (v, A^* g)_x)\, dx \text{ for all } g \in C^\infty(X, F),\ v \in C^\infty(X, E);$$

here Λ^p is the bundle of the exterior differential forms of the degree $0 \le p \le n$ over X. The Green operator always exists (see [15, Proposition 2.4.4]) and for the first order operator A it may be locally written in the following form:

$$G_A(\star g, v) = g^*(x)\ \sigma(A)(x, (\star dx_1, \dots, \star dx_n))\ v(x) \text{ for all } g \in C^\infty(X, F),\ v \in C^\infty(X, E).$$

Then it follows from Stokes formula that the (first) Green formula holds true:

$$\int_{\partial D} G_A(\star g, v) = \int_D ((Av, g)_x - (v, A^* g)_x)\ dx \text{ for all } g \in C^\infty(X, F),\ v \in C^\infty(X, E). \quad (1)$$

Fix a defining function of the domain D, i.e. a real valued C^∞-smooth function ρ with $|\nabla \rho| \neq 0$ on ∂D and such that $D = \{x \in X :\ \rho(x) < 0\}$. Without loss of a generality we can always choose the function ρ in such a way that $|\nabla \rho| = 1$ on a neighborhood of ∂D. Then

$$G_A(\star g, v) = \int_{\partial D} (\sigma(A)(x, \nabla \rho)\ v, g)_x\ \ ds(x) \text{ for all } g \in C^\infty(X, F),\ v \in C^\infty(X, E), \quad (2)$$

where ds is the volume form on ∂D induced from X.

Our principal object to study will be a complex $\{A_i, E_i\}_{i=0}^N$ of partial differential operators over X (see, [15], [14]),

$$0 \to C^\infty(X, E_0) \overset{A_0}{\to} C^\infty(X, E_1) \overset{A_1}{\to} C^\infty(X, E_2) \to \cdots \overset{A_{N-1}}{\to} C^\infty(X, E_N) \to 0, \quad (3)$$

where E_i are the bundles over X and $A_i \in \mathrm{Diff}_1(X; E_i \to E_{i+1})$ with $A_{i+1} \circ A_i \equiv 0$; we tacitly assume that $A_i = 0$ for both $i < 0$ and $i \ge N$. Obviously, $\sigma(A_{i+1}) \circ \sigma(A_i) \equiv 0$. We say that the complex $\{A_i, E_i\}_{i=0}^N$ is elliptic if the corresponding symbolic complex,

$$0 \to \pi^* E_0 \overset{\sigma(A_0)}{\to} \pi^* E_1 \overset{\sigma(A_1)}{\to} \pi^* E_2 \to \cdots \overset{\sigma(A_{N-1})}{\to} \pi^* E_N \to 0, \quad (4)$$

is exact for all $(x, z) \in T^* X \setminus \{0\}$, i.e. the the range of the map $\sigma(A_i)$ coincides with the kernel of the map $\sigma(A_{i+1})$. In particular, $\sigma(A_0)$ is injective away from the zero section of $T^* X$ and $\sigma(A_{N-1})$ is surjective.

As any differential complex is homotopically equivalent to a first order complex, we will consider elliptic complexes of first order operators only. Hence it follows that the Laplacians $\Delta_i = A_i^* A_i + A_{i-1} A_{i-1}^*$ of the complex are elliptic differential operators of the second order and types $E_i \to E_i$ on X for $0 \le i \le N$.

1.2 Sobolev spaces

We write $L^2(D, E)$ for the Hilbert space of all the measurable sections of E over D with a scalar product $(u, v)_{L^2(D,E)} = \int_D (u, v)_x dx$. We also denote $H^s(D, E)$ the Sobolev space of the distribution sections of E over D, whose weak derivatives up to the order $s \in \mathbb{N}$ belong to $L^2(D, E)$. As usual, let $H^s_{loc}(D \cup \Gamma, E)$ be the set of sections in D belonging to $H^s(\sigma, E)$ for every measurable set σ in D with $\overline{\sigma} \subset D \cup \Gamma$.

Further, for non-integer positive s we define the Sobolev spaces $H^s(D, E)$ with the use of the proper interpolation procedure (see, for example, [2, §1.4.11]). In the local situation we can use other (equivalent) approaches. For instance, if $X \subset \mathbb{R}^n$ and the bundle E is trivial, we may we denote $H^{1/2}(D,E)$ the closure of $C^\infty(\overline{D},E)$ functions with respect to the norm (see [26]):

$$\|u\|_{H^{1/2}(D,E)} = \sqrt{\|u\|^2_{L^2(D,E)} + \int_D \int_D \frac{|u(x)-u(y)|^2 dx\, dy}{|x-y|^{2n+1}}}.$$

Then, for $s \in \mathbb{N}$, let $H^{s-1/2}(D,E)$ be the space of functions from $H^{s-1}(D,E)$ such that the weak derivatives of the order $(s-1)$ belong to $H^{1/2}(D,E)$.

The Sobolev spaces of negative smoothness are usually defined with the use of a proper duality (see [27]). For instance, one can consider the Sobolev space $\tilde{H}^{-s}(D,E)$ as the completion of the space $C^\infty_{comp}(D,E)$ with respect to the norm $\sup\limits_{v\in C^\infty_{comp}(D,E)} \frac{|(u,v)_{L^2(D,E)}|}{\|v\|_{H^s(D,E)}}$, $s \in \mathbb{N}$. Unfortunately, elements of these spaces may have "bad" behavior near ∂D, but the study of the Cauchy problem needs a correctly defined notion of a trace. This is the reason we use slightly different spaces; we follow [22] (cf. [24], [2, Chapters 1, 9], [28]). More exactly, denote by $C^\infty_{m-1}(\overline{D},E)$ the subspace in $C^\infty(\overline{D},E)$ consisting of the sections with vanishing on ∂D derivatives up to order $m-1$. Let $s \in \mathbb{N}$. For sections $u \in C^\infty(\overline{D},E)$ we define two types of negative norms

$$\|u\|_{-s} = \sup_{v\in C^\infty(\overline{D},E)} \frac{|(u,v)_{L^2(D,E)}|}{\|v\|_{H^s(D,E)}}, \quad |u|_{-s} = \sup_{v\in C^\infty_{m-1}(\overline{D},E)} \frac{|(u,v)_{L^2(D,E)}|}{\|v\|_{H^s(D,E)}}.$$

It is more correct to write $\|\cdot\|_{-s,D,E}$ and $|\cdot|_{-s,D,E}$, but we prefer to omit the indexes D, E, if it does not cause misunderstandings. It is convenient to set $\|\cdot\|_{0,D} = \|\cdot\|_{L^2(D,E)}$.

Denote the completions of space $C^\infty(\overline{D},E)$ with respect to these norms by $H^{-s}(D,E)$ and $H(D,E,|\cdot|_{-s})$ respectively. It follows from the definition that the elements of these Banach spaces are distributions of finite orders on D and these spaces could be called the Sobolev spaces of negative smoothness. Clearly, they satisfy the following relations: $H^{-s}(D,E) \hookrightarrow H(D,E,|\cdot|_{-s}) \hookrightarrow \tilde{H}^{-s}(D,E)$, and, similarly, $H^{-s}(D,E) \hookrightarrow H^{-s-1}(D,E)$, $H(D,E,|\cdot|_{-s}) \hookrightarrow H(D,E,|\cdot|_{-s-1})$.

The Banach space $H^{-s}(D,E)$ can be identifyed with the dual space $(H^s(D,E))'$ of the standard Hilbert space $H^s(D,E)$ (see, for instance, [2, Theorem 1.4.28]).

Clearly, any element $u \in H^{-s}(D,E)$ extends up to an element $U \in H^{-s}(\overset{\circ}{X},E)$ via

$$\langle U, v\rangle_{\overset{\circ}{X}} = \langle u, v\rangle_D \text{ for all } v \in H^s(\overset{\circ}{X},E);$$

here $\langle\cdot,\cdot\rangle_D$ is a pairing $H \times H'$ for a space H of distributions over D. It is natural to denote this extension $\chi_D u$ because its support belongs to $\overline{D}$. Obviously, this extension induces a bounded linear operator

$$\chi_D : H^{-s}(D,E) \to H^{-s}(\overset{\circ}{X},E), \quad s \in \mathbb{Z}_+. \tag{5}$$

It is known that the differential operator A continuously maps $H^s(D,E)$ to $H^{s-m}(D,F)$, $m \le s$, $s \in \mathbb{N}$. The following lemma shows the specific way of the action of A for $s \le 0$.

Lemma 1.1. *A differential operator A induces linear bounded operator $A : H^{-s}(D,E) \to H(D,F,|\cdot|_{-s-m})$, $s \in \mathbb{Z}_+$.*

Proof. Immediately follows from (1) and (2). □

However there is no need for elements of $H^{-s}(D,E)$ to have a trace on ∂D and there is no need for A to map $H^{-s}(D,E)$ to $H^{-s-m}(D,F)$.

2 Traces of Sobolev functions of negative smoothness

By the discussion above we need to introduce some other spaces in order to define the traces on ∂D. In general, our approach is closed to the one described in [2, §9.2, 9.3].

2.1 Strong traces on the boundary

It is well known that if ∂D is sufficiently smooth then the functions from the Sobolev space $H^s(D)$, $s \in \mathbb{N}$, have traces on the boundary in the Sobolev space $H^{s-1/2}(\partial D)$ and the corresponding trace operator $t_s : H^s(D) \to H^{s-1/2}(\partial D)$ is bounded and it admits the bounded right inverse operator (see, for instance, [26]). In particular, this means that for every $u \in H^s_{loc}(D \cup \Gamma, E)$, $s \in \mathbb{N}$, there is the trace $t_{\Gamma,E}(u)$ on Γ belonging to $H^{s-1/2}_{loc}(\Gamma, E)$.

In order to define the so-called strong traces on ∂D for elements of Sobolev spaces with negative smoothness we denote $H^{-s}_t(D,E)$ the completion of $C^\infty(\overline{D}, E)$ with respect to the graph-norm:

$$\|u\|_{-s,t} = \left(\|u\|^2_{-s} + \|u\|^2_{-s-1/2,\partial D}\right)^{1/2}.$$

Thus the operator t_s induces the bounded linear trace operator

$$t_{-s,E} : H^{-s}_t(D,E) \to H^{-s-1/2}(\partial D, E).$$

Remark 2.1. The spaces $H^{-s}(D,E)$, $H^{-s}_t(D,E)$, $H(D,E,|\cdot|_{-s})$ are well known. Let A be a first order operator with injective principal symbol. Given distributions w and u_0, consider the Dirichlet problem for strongly elliptic formally self-adjoint second order operator A^*A. It consists in finding a distribution u satisfying

$$\begin{cases} A^*Au &= w \quad \text{in} \quad D, \\ t(u) &= u_0 \quad \text{on} \quad \partial D. \end{cases} \tag{6}$$

It follows from [22, theorems 2.1 and 2.2] (see also [24], [28] for systems of equations) that the Uniqueness Theorem and the Existence Theorem are valid for problem (6) on the Sobolev scale $H^s(D,E)$, $s \in \mathbb{Z}$ for data $w \in H(D,E,|\cdot|_{s-2})$ and $u_0 \in H^{s-1/2}(\partial D, E)$. Denote by $\mathcal{P}^{(D)}$ the operator mapping u_0 and $w = 0$ to the unique solution to the Dirichlet problem (6). Similarly, denote $\mathcal{G}^{(D)}$ the operator mapping w to the unique solution to the Dirichlet problem (6) with zero boundary Dirichlet data. Clearly, $\mathcal{G}^{(D)}_{A^*A}$ is the famous Green function of the Dirichlet problem (6) and $\mathcal{P}^{(D)}_{A^*A}$ is the Poisson integral corresponding to the problem. The standard theorem of improving the smoothness of the Dirichlet problem (see, for instance, [26] or [2, Theorem 9.3.17]) and [28, Theorem 2.26 and Corollary 2.31] imply that the operators $\mathcal{P}^{(D)}_{A^*A}$, $\mathcal{G}^{(D)}_{A^*A}$ act continuously on the following Sobolev scale:

$$\mathcal{P}^{(D,s)}_{A^*A} : H^{s-1/2}(\partial D, E) \to H^s(D,E), \quad \mathcal{G}^{(D,s)}_{s,A^*A} : H(D,E,|\cdot|_{s-2}) \to H^s(D,E), \quad s \in \mathbb{N},$$

$$\mathcal{P}^{(D,-s)}_{A^*A} : H^{-s-1/2}(\partial D, E) \to H^{-s}_t(D, E), \quad \mathcal{G}^{(D,-s)}_{A^*A} : H(D, E, |\cdot|_{-s-2}) \to H^{-s}_t(D, E), \quad s \in \mathbb{Z}_+.$$

They completely describe the solutions of the Dirichlet problem on the scale of the Sobolev spaces. □

However we need a more subtle characteristic of the traces to study the Cauchy problem for the differential complex $\{A_i\}$.

For a section u of E over D and a first order operator A, let $\tilde{\tau}_A(u) = \sigma(A)(x, \nabla\rho(x))u$ represent the Cauchy data of u with respect to A (see, for instance, [15, §3.2.2]). Similarly, let $\tilde{\nu}_A(f) = \tilde{\tau}_{A^*}(f)$ represent the Cauchy data of f with respect to A^* for a section f of F. Then the maps $\tilde{\tau}$, $\tilde{\nu}$ induces a bounded linear operators

$$\tilde{\tau}_{A,s} : H^s(D, E) \to H^{s-1/2}(\partial D, F), \quad \tilde{\nu}_{A,s} : H^s(D, F) \to H^{s-1/2}(\partial D, E), \quad s \in \mathbb{N}. \tag{7}$$

Denote the completions of the space $C^\infty(\overline{D}, E)$ with respect to graph-norms

$$\|u\|_{-s,A} = \left(\|u\|^2_{-s} + \|Au\|^2_{-s-1}\right)^{1/2}, \quad \|u\|_{-s,\tilde{\tau}_A} = \left(\|u\|^2_{-s} + \|\tilde{\tau}_A(u)\|^2_{-s-1/2,\partial D}\right)^{1/2}$$

by $H^{-s}_A(D, E)$ and $H^{-s}_{\tilde{\tau}_A}(D, E)$ respectively. Clearly, the elements of these spaces are more regular in $\overline{D}$ than the elements of $H^{-s}(D, E)$. Moreover, by the very definition, the differential operator A induces a bounded linear operator

$$A_{-s} : H^{-s}_A(D, E) \to H^{-s-1}(D, F),$$

and the trace operator (7) induces a bounded linear operator

$$\tilde{\tau}_{A,s} : H^{-s}_{\tilde{\tau}_A}(D, E) \to H^{-s-1/2}(\partial D, F).$$

Theorem 2.1. *The linear spaces $H^{-s}_A(D, E)$ and $H^{-s}_{\tilde{\tau}_A}(D, E)$ coincide and their norms are equivalent. Moreover, if A has the injective principal symbol then the spaces $H^{-s}_{\tilde{\tau}_A}(D, E)$ and $H^{-s}_t(D, E)$ coincide and their norms are equivalent.*

Proof. It follows from the definition of the spaces that we need to check the relations between the norms on the sections from $C^\infty(\overline{D}, E)$ only. By Green's formula (1) and (2) we have for all $u \in C^\infty(\overline{D}, E)$:

$$\|u\|^2_{-s,A} \le (1 + \|A^*_{s+1}\|^2 + \|t_{s+1,F}\|^2)(\|u\|^2_{-s} + \|\tilde{\tau}_A(u)\|^2_{-s-1/2,\partial D}),$$

where $A^*_{s+1} : H^{s+1}(D, F) \to H^s(D, F)$ is the linear bounded operator induced by the differential operator A^*.

Back, fix a section $g_0 \in C^\infty(\partial D, F)$. Now let $\nabla_F \in \mathrm{Diff}_1(X; F \to F \otimes (T^*X)_c)$ be a connections in the bundle F compatible with the corresponding Hermitian metric (see [29, Ch. III, Proposition 1.11]). Obviously ∇_F has the injective symbol. Then, using remark 2.1 we see that there is a section $g \in C^\infty(\overline{D}, F)$ with $g = g_0$ on ∂D and $\|g\|_{s+1} \le \gamma\|g_0\|_{s+1/2}$. For instance we may take $g = \mathcal{P}^{(D)}_{\nabla^*_F \nabla_F} g_0$. Therefore Green's formula (1) and formula (2) imply that for all $u \in C^\infty(\overline{D}, E)$ we have:

$$\int_{\partial D} (\tilde{\tau}_A(u), g_0)_x ds(x) = \int_D ((Au, g)_x - (u, A^*g)_x) dx.$$

Hence
$$\|u\|^2_{-s} + \|\tilde\tau_A(u)\|^2_{-s-1/2,\partial D} \leq (1+\gamma^2\|A^*_{s+1}\|^2+\gamma^2)(\|u\|^2_{-s}+\|Au\|^2_{-s-1}),$$
i.e. the spaces $H^{-s}_A(D,E)$ and $H^{-s}_{\tilde\tau_A}(D,E)$ coincide and their norms are equivalent.

Finally, if the symbol $\sigma(A)$ is injective then the map $\sigma^*(A)(x,\nabla\rho(x))\sigma(A)(x,\nabla\rho(x))$ is invertible on ∂D and

$$\tilde\tau_A(u) = \sigma(A)(x,\nabla\rho(x))t(u), \quad t(u) = (\sigma^*(A)(x,\nabla\rho(x))\sigma(A)(x,\nabla\rho(x)))^{-1}\tilde\nu_A(\tilde\tau_A(u)),$$

which means that the norms $\|\cdot\|_{-s,t}$ and $\|\cdot\|_{-s,\tilde\tau_A}$ are equivalent on $C^\infty(\overline{D},E)$. □

Now for the complex $\{A_i\}$ denote $\tilde\tau_i$ the Cauchy data with respect to A_i. Similarly denote $\tilde\nu_i$ the Cauchy data with respect to A^*_{i-1}. As the complex is elliptic then the matrix $L(x) = \sigma^*(A_i)(x,\nabla\rho(x))\sigma(A_i)(x,\nabla\rho(x)) + \sigma(A_{i-1})(x,\nabla\rho(x))\sigma^*(A_{i-1})(x,\nabla\rho(x))$ is invertible in a neighborhood of ∂D. Then we set

$$\tau_i = L^{-1}(x)\tilde\nu_{i+1}\circ\tilde\tau_i, \quad \nu_i = L^{-1}(x)\tilde\tau_{i-1}\circ\tilde\nu_i.$$

Lemma 2.1. *The following identities hold true:*

$$\tilde\tau_{i+1}\circ\tilde\tau_i = 0,\ \tilde\nu_{i-1}\circ\tilde\nu_i = 0,\ \tilde\tau_i\circ\nu_i = 0,\ \tilde\nu_i\circ\tau_i = 0,\ \tilde\tau_i = \tilde\tau_i\circ\tau_i,\ \tilde\nu_i = \tilde\nu_i\circ\nu_i,$$

$$\tau_i\circ\tau_i = \tau_i,\ \nu_i\circ\nu_i = \nu_i,\ \tau_i\circ\nu_i = 0,\ \nu_i\circ\tau_i = 0,\ \tau_i+\nu_i = 1,$$
$$\tau_i^* = \tau_i,\ \nu_i^* = \nu_i,\ \tilde\tau_i^* = \tilde\nu_{i+1},\ \tilde\nu_i^* = \tilde\tau_{i-1}.$$

Proof. See, for instance, [15, formulae (3.2.3)]. □

Because of Lemma 2.1, the projections $\tau_i(u)$ and $\nu_i(u)$ are often called the tangential and the normal parts of a section u with respect to the complex $\{A_i\}$ respectively.

Due to Lemma 2.1 we have for all $u\in C^\infty(\overline{D},E_i)$, $g\in C^\infty(\overline{D},E_{i+1})$:

$$\int_{\partial D}(\tau_i(u),\tilde\nu_{i+1}(g))_x\,ds(x) = \int_D((A_iu,g)_x-(u,A_i^*g)_x)dx. \tag{8}$$

Denote the completion of the space $C^\infty(\overline{D},E_i)$ $(0\leq i\leq N)$ with respect to graph-norms

$$\|u\|_{-s,\tau_i} = \left(\|u\|^2_{-s}+\|\tau_i(u)\|^2_{-s-1/2,\partial D}\right)^{1/2}, \quad \|u\|_{-s,\nu_i} = \left(\|u\|^2_{-s}+\|\nu_i(u)\|^2_{-s-1/2,\partial D}\right)^{1/2}$$

by $H^{-s}_{\tau_i}(D,E_i)$ and $H^{-s}_{\nu_i}(D,E_i)$ respectively.

Corollary 2.1. *Let the differential complex $\{A_i\}$ be elliptic. Then the linear spaces $H^{-s}_{A_i}(D,E_i)$, $H^{-s}_{\tilde\tau_i}(D,E_i)$ and $H^{-s}_{\tau_i}(D,E_i)$ coincide and their norms are equivalent.*

Proof. The equivalence of the norms $\|\cdot\|_{-s,A_i}$ and $\|\cdot\|_{-s,\tilde\tau_i}$ follows Theorem 2.1. Finally, as the complex $\{A_i\}$ is elliptic then Lemma 2.1 implies the equivalence of the norms $\|\cdot\|_{-s,\tilde\tau_i}$ and $\|\cdot\|_{-s,\tau_i}$. □

Corollary 2.2. *Let the complex $\{A_i\}$ be elliptic. Then linear spaces $H^{-s}_{A^*_{i-1}}(D,E_i)$, $H^{-s}_{\tilde\nu_i}(D,E_i)$ and $H^{-s}_{\nu_i}(D,E_i)$ coincide and their norms are equivalent.*

Proof. As the complex $\{A_i\}$ is elliptic then the complex $\{A_i^*\}$ is elliptic too. That is why Corollary 2.1 implies the desired statement. $\square$

Corollary 2.3. *If the complex $\{A_i\}$ is elliptic then the linear spaces $H^{-s}_{A_i \oplus A^*_{i-1}}(D, E_i)$ and $H^{-s}_t(D, E_i)$ coincide and their norms are equivalent.*

Proof. As the complex $\{A_i\}$ is elliptic then the operator $A_i \oplus A^*_{i-1}$ has the injective principal symbol. Hence the statement follows from Theorem 2.1. $\square$

Corollary 2.4. *If the complex $\{A_i\}$ is elliptic then the following identities hold true:*

$$H^{-s}_{A_i \oplus A^*_{i-1}}(D, E_i) = H^{-s}_{A_i}(D, E_i) \cap H^{-s}_{A^*_{i-1}}(D, E_i),$$

$$H^{-s}_t(D, E_i) = H^{-s}_{\tau_i}(D, E_i) \cap H^{-s}_{\nu_i}(D, E_i).$$

2.2 Weak boundary values of the tangential and normal parts

Consider now the weak extension of an operator A on the scale $H^{-s}(D, E)$. Namely, denote $H^s_{A,w}(D, E)$ the set of the sections u from $H^{-s}(D, E)$ such that there is a section $f \in H^{-s}(D, F)$ satisfying $Au = f$ in $H^{-s}(D, F, |\cdot|_{-s-1})$ (in particular, in the sense of distributions in D). As the operator A is linear, this set is linear too. Clearly,

$$H^{-s}_A(D, E) \subset H^{-s}_{A,w}(D, E). \tag{9}$$

It is natural to expect that these spaces coincide (cf. [30]); we will prove it later.

According to Corollary 2.1, we have $\tau_i(u) \in H^{-s-1/2}(\partial D, E_i)$ for all sections $u \in H^{-s}_A(D, E_i)$. Let us clarify the situation with the traces of the elements from $H^{-s}_{A_i,w}(D, E_i)$ for an operator A_i from an elliptic complex.

To this end, define pairing (u, v) for $u \in H^{-s}(D, E)$, $v \in C^\infty(\overline{D}, E)$ as follows. By the definition, one can find such a sequence $\{u_\nu\}$ in $C^\infty(\overline{D}, E)$ that $\|u_\nu - u\|_{-s} \to 0$ if $\nu \to \infty$. Then

$$|(u_\nu - u_\mu, v)_{L^2(D,E)}| \leq \|u_\nu - u_\mu\|_{-s} \|v\|_{H^s(D,E)} \to 0 \text{ as } \mu, \nu \to \infty.$$

Set $(u, v) = \lim\limits_{\nu \to \infty} (u_\nu, v)_{L^2(D,E)}$. It is clear that the limit does not depend on the choice of the sequence $\{u_\nu\}$, for if $\|u_\nu\|_{-s} \to 0$, $\nu \to \infty$, then $|(u_\nu, v)_{L^2(D,E)}| \leq \|u_\nu\|_{-s} \|v\|_{H^s(D,E)}$ tends to zero too. This implies that for $u \in H^{-s}(D, E)$ and $v \in C^\infty(\overline{D}, E)$ we have the inequality: $|(u, v)| \leq \|u\|_{-s} \|v\|_{H^s(D,E)}$. Set $H(D, E) = \cup_{s=0}^\infty H^{-s}(D, E)$. Easily, the pairing $(u, v)_D$ is correctly defined for $u \in H(D, E)$ and $v \in C^\infty(\overline{D}, E)$. The unions $\cup_{s=1}^\infty H^{-s}(D, E)$ and $\cup_{s=1}^\infty H^{-s}_{A,w}(D, E)$ we denote by $H(D, E)$ and $H_A(D, E)$ respectively.

As before, let Γ be an open (in the topology of ∂D) connected subset of ∂D. The following definition is induced by (8).

Definition 2.1. *Let alone the correctness of this definition, we say that a distribution-section $u \in H_{A_i}(D, E_i)$, satisfying $A_i u = f$ in D with $f \in H(D, E_{i+1})$, has a weak boundary value $\tau^w_{i,\Gamma}(u) = \tau_i(u_0)$ on Γ for $u_0 \in \mathcal{D}'(\Gamma, E_i)$ if*

$$(f, g)_D - (u, A_i^* g)_D = \langle \star\tilde{\nu}_{i+1}(g), \tau_i(u_0) \rangle_\Gamma \textit{ for all } g \in C^\infty_{comp}(D \cup \Gamma, E_{i+1}).$$

Formulae (1), (2) and Theorem 2.1 imply that any section $u \in H^{-s}_{A_i}(D, E_i)$ has a weak boundary value of the tangential part $\tau^w_{i,\partial D}(u)$ on ∂D coinciding with the trace $\tau_{i,-s}(u) \in H^{-s-1/2}(\partial D, E_i)$. We are to connect the weak boundary values of the tangential parts with the so-called *limit boundary values* of the solutions of finite orders of growth near ∂D to elliptic systems (see [23], [24], [2]). Recall that a solution $u \in S_A(D)$ of an elliptic system A has a finite order of growth near ∂D if for any point $x^0 \in \partial D$ there are a ball $B(x^0, R)$ and constants $c > 0$, $\gamma > 0$ such that

$$|v(x)| \leq c\ dist(x, \partial D)^{-\gamma} \text{ for all } x \in B(x^0, R) \cap D.$$

As ∂D is compact, the constants c and γ may be chosen in such a way that this estimate is valid for all $x^0 \in \partial D$. The space of all the solutions to A of finite order of growth near ∂D will be denoted $S^F_A(D)$.

Further, set $D_\varepsilon = \{x \in D : \rho(x) < -\varepsilon\}$. Then, for sufficiently small $\varepsilon > 0$, the sets $D_\varepsilon \Subset D \Subset D_{-\varepsilon}$ are domains with smooth boundaries $\partial D_{\pm\varepsilon}$ of class C^∞. Besides, the vectors $\mp\varepsilon\nu(x)$ belong to $\partial D_{\pm\varepsilon}$ for every $x \in \partial D$ (here $\nu(x)$ is the external normal unit vector to the hypersurface ∂D at the point x). According to [2, Theorem 9.4.7], [24], if A is elliptic and it satisfies the Uniqueness Condition 1.1 then any solution $w \in S^F_{A^*A}(D)$ had the weak limit value $w^0 \in (C^\infty_{comp}(\Gamma, E))'$ on Γ, i.e.

$$< w^0, v >= \lim_{\varepsilon \to +0} \int_{\partial D} v(y) w(y - \varepsilon\nu(y)) ds(y) \text{ for all } v \in C^\infty_{comp}(\Gamma, E).$$

Theorem 2.2. *Let A_i be an elliptic complex such that the operators $A_i \oplus A^*_{i-1}$, $0 \leq i \leq N$, satisfy the Uniqueness Condition 1.1. Then every section $u \in H^{-s}_{A_i,w}(D, E_i)$ has the weak boundary value $\tau^w_{i,\partial D}(u) \in H^{-s-1/2}(\partial D, E_i)$ in the sense of Definition 2.1, coinciding with the limit boundary value $\tau_i(w)$ of the solution $w = (u - \mathcal{G}^{(D,-s)}_{\Delta_i} A^*_i f - A_{i-1}\mathcal{G}^{(D,-s+1)}_{\Delta_{i-1}} A^*_{i-1} u)$ from $S^F_{\Delta_i}(D)$; besides, $\tau^w_{i,\partial D}(u)$ does not depend on the choice of $f \in H^{-s-1}(D, E_{i+1})$ with $A_i u = f$ in D.*

Proof. First of all we note that Lemma 1.1, Theorem 2.1 and Remark 2.1, imply that the operator $\mathcal{G}^{(D,p)}_{\Delta_i} A^*_i$ continuously maps $H^{p-1}(D, E_{i+1})$ to $H^p_{A_i}(D, E_i)$. Hence the sections $w_1 = \mathcal{G}^{(D,-s)}_{\Delta_i} A^*_i f \in H^{-s}_{A_i}(D, E_i)$ and $w_2 = \mathcal{G}^{(D,-s+1)}_{\Delta_{i-1}} A^*_{i-1} u \in H^{-s+1}_{A_{i-1}}(D, E_{i-1})$ have the zero traces $t_{-s}(w_1)$ and $t_{-s+1}(w_2)$ on ∂D. In particular, $\tau_{i,-s}(w_1) = 0$, $\tau_{i-1,-s+1}(w_2) = 0$, and therefore $\tau^w_{i,\partial D}(w_1) = 0$, $\tau^w_{i-1,\partial D}(w_2) = 0$. Besides, as $A_i \circ A_{i-1} \equiv 0$, we see that $A_i(A_{i-1}w_2) = 0$ in D and $A_{i-1}w_2 \in H^{-s}_{A_i,w}(D, E_i)$. According to Definition 2.1, applied to w_2, we have:

$$(0, \psi)_D - (A_{i-1}w_2, A^*_i v)_D = -\langle \star\tilde{\nu}_i(A^*_i v), \tau_{i-1}(w_2)\rangle_\Gamma + (w_2, A^*_{i-1}A^*_i v)_D = 0$$

for all $v \in C^\infty_{comp}(D \cup \Gamma, E_{i+1})$. Therefore $\tau^w_{i,\partial D}(A_{i-1}w_2) = 0$ too.

It is clear now that the section $u \in H^{-s}_{A_i,w}(D, E_i)$ has the weak boundary value of $\tau^w_{i,\partial D}(u)$ in the sense of Definition 2.1 if and only if the section $w = (u - \mathcal{G}^{(D,-s)}_{\Delta_i} A^*_i f - A_{i-1}\mathcal{G}^{(D,-s+1)}_{\Delta_{i-1}} A^*_{i-1} u)$ has. By the construction $w \in H^{-s}_{A_i,w}(D, E_i)$ satisfies

$$\Delta_i w = (A^*_i A_i + A_{i-1}A^*_{i-1})u - A^*_i f - A_{i-1}(A^*_{i-1}u) = 0 \text{ in } D.$$

In particular, this section belongs to $C^\infty(D, E_i)$, it has a finite order of growth near ∂D (see [28, Theorem 2.32]), and hence it has the limit boundary value $w^0 \in (C^\infty_{comp}(\partial D, E_i))'$ on ∂D (see [2, Theorem 9.4.8]). Of course, the section $\tau_i(w^0) \in (C^\infty_{comp}(\partial D, E_i))'$ is also defined because the function ρ is of class C^∞. Clearly, $\tau_i(w) = \tau_i(w^0)$ in the sense of the limit boundary values on ∂D.

As we have already noted, $w \in H^{-s}_{A_i,w}(D, E_i)$ and $A_i w = f - A_i \mathcal{G}^{(D,-s)}_{\Delta_i} A_i^* f$ in D where $(f - A_i \mathcal{G}^{(D,-s)}_{\Delta_i} A_i^* f) \in H^{-s-1}(D, E_{i+1})$. In particular, this means that

$$\langle \chi_D w, v\rangle = (w, v)_D \text{ for all } v \in C^\infty(\overset{\circ}{X}, E_i),$$

$$\langle \chi_D (f - A_i \mathcal{G}^{(D,-s)}_{\Delta_i} A_i^* f), g\rangle = (f - A_i \mathcal{G}^{(D,-s)}_{\Delta_i} A_i^* f, g)_D \text{ for all } g \in C^\infty(\overset{\circ}{X}, E_{i+1}).$$

Since the both w and $A_i w$ are solutions to elliptic operators, i.e. $\Delta_i w = 0$ in D, $\Delta_{i+1}(A_i w) = 0$ in D and they both have finite orders of growth near ∂D, then it follows from [2, the proof of Theorem 9.4.7] that there is a sequence of positive numbers $\{\varepsilon_\nu\}$, tending to zero and such that

$$\langle \chi_D w, v\rangle = \lim_{\varepsilon_\nu \to +0} \int_{D_{\varepsilon_\nu}} (w, v)_x dx \text{ for all } v \in C^\infty(\overset{\circ}{X}, E_i),$$

$$\langle \chi_D (f - A_i \mathcal{G}^{(D,-s)}_{\Delta_i} A_i^* f), g\rangle = \lim_{\varepsilon_\nu \to +0} \int_{D_{\varepsilon_\nu}} (A_i w, g)_x dx \text{ for all } g \in C^\infty(\overset{\circ}{X}, E_{i+1}).$$

By Whitney's Theorem, every smooth section over $\overline{D}$ may be extended up to a smooth section over X. Therefore

$$(w, v)_D = \lim_{\varepsilon_\nu \to +0} \int_{D_{\varepsilon_\nu}} (w, v)_x dx \text{ for all } v \in C^\infty(\overline{D}, E_i),$$

$$(f - A_i \mathcal{G}^{(D,-s)}_{\Delta_i} A_i^* f, g)_D = \lim_{\varepsilon_\nu \to +0} \int_{D_{\varepsilon_\nu}} (A_i w, g)_x dx \text{ for all } g \in C^\infty(\overline{D}, E_{i+1}).$$

As $\tau_i(\mathcal{G}^{(D,-s)}_{\Delta_i} A_i^* f + A_{i-1} \mathcal{G}^{(D,-s+1)}_{\Delta_{i-1}} A_i^* u) = 0$ on ∂D in the sense of Definition 2.1, we see that Lemma 2.1, formulae (1) and (8) imply for all $g \in C^\infty(\overline{D}, E_{i+1})$:

$$(f, g)_D - (u, A_i^* g)_D = (f - A_i \mathcal{G}^{(D,-s)}_{\Delta_i} A_i^* f, g)_D - (w, A_i^* g)_D =$$

$$\lim_{\varepsilon_\nu \to +0} \left(\int_{D_{\varepsilon_\nu}} ((A_i w, g)_x - (w, A_i^* g)_x) dx \right) =$$

$$\lim_{\varepsilon_\nu \to +0} \int_{\partial D_{\varepsilon_\nu}} (\tau_i(w), \tilde{\nu}_{i+1}(g))_x \, ds(x) = \langle \star \tilde{\nu}_{i+1}(g), \tau_i(w^0)\rangle_{\partial D},$$

i.e. $\tau^w_{i,\partial D}(u) = \tau_i(w)$ on ∂D. Now, if $\tilde{f} \in H^{-s-1}(D, E_{i-1})$ satisfies $A_i u = \tilde{f}$ in D then $\tilde{w} = (u - \mathcal{G}^{(D,-s)}_{\Delta_i} A_i^* \tilde{f} - A_{i-1} \mathcal{G}^{(D,-s+1)}_{\Delta_{i-1}} A_{i-1}^* u)$ and we have: $(w - \tilde{w}) = \mathcal{G}^{(D,-s)}_{\Delta_i} A_i^* (f - \tilde{f}) \in H^{-s}_{A_i}(D, E_i)$ with $\tau^w_{i,\partial D}(w - \tilde{w}) = 0$ on ∂D, i.e. the weak boundary value $\tau^w_{i,\partial D}(u)$ does not depend on the choice of the section $f \in H^{-s-1}(D, E_{i+1})$ satisfying $A_i u = f$ in D.

Finally, we are to prove that the weak boundary value belongs to the corresponding Sobolev space $H^{-s-1/2}(\partial D, E_i)$. With this aim, fix a section $v_0 \in C^\infty(\partial D, E_{i+1})$. Then the section

$g = \mathcal{P}^{(D)}_{\nabla^*_{E_{i+1}}\nabla_{E_{i+1}}}\tilde{\tau}_i(v_0)$ (see the proof of Theorem 2.1) belongs to $C^\infty(\overline{D}, E_{i+1})$ and coincides with $\tilde{\tau}_i(v_0)$ on ∂D. Moreover, according to Remark 2.1 we have:

$$\|g\|_{H^{s+1}(D,E_{i+1})} \leq \gamma_1 \|\tilde{\tau}_i(v_0)\|_{H^{s+1/2}(\partial D,E_{i+1})} \leq \gamma_2 \|v_0\|_{H^{s+1/2}(\partial D,E_i)} \tag{10}$$

with a positive constants γ_1, γ_2, which does not depend on g and v_0. Hence, by Definition 2.1 and Lemma 2.1, we obtain:

$$|(\tau^w_{i,\partial D}(u), v_0)_{\partial D}| = |\langle \star\tilde{\nu}_{i+1}(\tilde{\tau}_i(v_0)), \tau^w_{i,\partial D}(u)\rangle_{\partial D}| = |\langle \star\tilde{\nu}_{i+1}(g), \tau^w_{i,\partial D}(u)\rangle_{\partial D}| =$$

$$|(f,g)_D - (u, A_i^* g)_D| \leq \|f\|_{-s-1}\|g\|_{H^{s+1}(D,E_{i+1})} + \|u\|_{-s}\|A_i^* g\|_{H^s(D,E_i)}.$$

As the map $A_i^* : H^{s+1}(D, E_{i+1}) \to H^s(D, E_i)$ is bounded, then the estimate implies that (10)

$$|(\tau^w_{i,\partial D}(u), v_0)| \leq \tilde{\gamma}(\|u\|_{-s} + \|f\|_{-s-1})\|v_0\|_{H^{s+1/2}(\partial D,E_{i+1})}$$

with a positive constant $\tilde{\gamma}$ which does not depend on v_0 and u_0.

Hence,

$$\|\tau^w_{i,\partial D}(u)\|_{H^{-s-1/2}(\partial D,E_i)} = \sup_{v\in C^\infty_{comp}(\partial D,E_i)} \frac{|(\tau^w_{i,\partial D}(u), v)_{\partial D}|}{\|v\|_{H^{s+1/2}(\partial D,E_i)}} \leq \tilde{\gamma}(\|u\|_{-s} + \|f\|_{-s-1}).$$

Thus, the section $\tau^w_{i,\partial D}(u)$ belongs to the space $H^{-s-1/2}(\partial D, E_i)$, which was to be proved. □

Corollary 2.5. *The spaces $H^{-s}_{A_i}(D, E_i)$ and $H^{-s}_{A_i,w}(D, E_i)$ coincide.*

Proof. Since (9), it is enough to prove that $H^{-s}_{A_i,w}(D, E_i) \subset H^{-s}_{A_i}(D, E_i)$. Fix a section $u \in H^{-s}_{A_i,w}(D, E_i)$. Proving Theorem 2.2 we have seen that there is $w \in S^F_{\Delta_i}(D) \cap H^{-s}(D, E_i)$, satisfying

$$u = w + \mathcal{G}^{(D,-s)}_{\Delta_i} A_i^* f + A_{i-1}\mathcal{G}^{(D,-s+1)}_{\Delta_{i-1}} A^*_{i-1} u.$$

According to Remark 2.1, the section w is presented via its boundary values on ∂D by the Poisson type integral $w = \mathcal{P}^{(D,-s)}_{\Delta_i} t_i(w)$. Hence $w \in H^{-s}_t(D, E_i)$. Besides, Remark 2.1 imply that $w_1 = \mathcal{G}^{(D,-s)}_{\Delta_i} A_i^* f$ belongs to $H^{-s}_t(D, E_i)$ too. Thus, it follows from Corollary 2.3 that the sections w and w_1 belong $H^{-s}_{A_i\oplus A^*_{i-1}}(D, E_i) \subset H^{-s}_{A_i}(D, E_i)$.

Take a sequence $\{u_\nu\} \subset C^\infty(\overline{D}, E_i)$ approximating u in the space $H^{-s}(D, E_i)$. It follows from Remark 2.1 and 1.1 that the sequence $\{A_{i-1}\mathcal{G}^{(D,-s+1)}_{\Delta_{i-1}} A^*_{i-1} u_\nu\} \subset C^\infty(\overline{D}, E_i)$ converges to $A_{i-1}\mathcal{G}^{(D,-s+1)}_{\Delta_{i-1}} A^*_{i-1} u$ in the space $H^{-s}(D, E_i)$. Moreover, $\{A_i(A_{i-1}\mathcal{G}^{(D,-s+1)}_{\Delta_{i-1}} A^*_{i-1} u_\nu) \equiv 0\} \subset C^\infty(\overline{D}, E_i)$ converges to zero in the space $H^{-s-1}(D, E_{i+1})$. Therefore $A_{i-1}\mathcal{G}^{(D,-s+1)}_{\Delta_{i-1}} A^*_{i-1} u$ belongs to $H^{-s}_{A_i}(D, E_i)$. That is why the section u belongs to this space too. □

Corollary 2.6. *The differential operator A_i continuously maps $H^{-s}_{A_i}(D, E_i)$ to $H^{-s-1}_{A_{i+1}}(D, E_{i+1})$.*

Similarly defining the spaces $H^{-s}_{A^*_{i-1},w}(D, E_i)$ and $H^{-s}_{A_i\oplus A^*_{i-1},w}(D, E_i)$ we easily obtain the following statements.

Corollary 2.7. *The spaces $H^{-s}_{A^*_{i-1}}(D, E_i)$ and $H^{-s}_{A^*_{i-1},w}(D, E_i)$ coincide.*

Corollary 2.8. *The spaces $H^{-s}_{A_i \oplus A^*_{i-1}}(D, E_i)$ and $H^{-s}_{A_i \oplus A^*_{i-1},w}(D, E_i)$ coincide.*

As we have seen above, the scale $\{H^{-s}_{A_i}(D, E_i)\}$ is suitable for stating the Cauchy problem for the elliptic first order complex $\{A_i\}$. In order to do this we need to choose a proper spaces for the boundary Cauchy data on a surface $\Gamma \subset \partial D$. As we are interesting in the case $\Gamma \neq \partial D$, we will use one more type of the Sobolev spaces: the Sobolev spaces on closed sets (see, for instance, [2, §1.1.3]). Namely, let $H^{-s-1/2}(\overline{\Gamma}, E_i)$ stand for the factor space of $H^{-s-1/2}(\partial D, E_i)$ over the subspace of functions vanishing on a neighborhood of $\overline{\Gamma}$. Of course, it is not so easy to handle this space, but its every element extends from Γ up to an element of $H^{-s-1/2}(\partial D, E_i)$. Further characteristic of this space may be found in [2, Lemma 12.3.2]). We only note that if Γ has C^∞-smooth boundary (on ∂D), then

$$H^{-s-1/2}(\Gamma, E_i) \hookrightarrow H^{-s-1/2}(\overline{\Gamma}, E_i) \hookrightarrow \tilde{H}^{-s-1/2}(\Gamma, E_i).$$

Corollary 2.9. *For every section $u \in H^{-s}_{A_i}(D, E_i)$ and every $\Gamma \subset \partial D$ there is the boundary value $\tau_{i,\Gamma}(u)$ in the sense of Definition 2.1, belonging to $H^{-s-1/2}(\overline{\Gamma}, E_i)$.*

As ∂D is compact, $\cup_{s=1}^\infty H^{-s-1/2}(\partial D, E_i) = \mathcal{D}'(\partial D, E_i)$. Set $\cup_{s=1}^\infty H^{-s-1/2}(\overline{\Gamma}, E_i) = \mathcal{D}'(\overline{\Gamma}, E_i)$. Now Corollary 2.5 immediately implies the following statements.

Corollary 2.10. *For every $u \in H_{A_i}(D, E_i)$ and every $\Gamma \subset \partial D$ there is the boundary value $\tau_{i,\Gamma}(u)$ in the sense of Definition 2.1, belonging to $\mathcal{D}'(\overline{\Gamma}, E_i)$.*

3 A homotopy formula

In this section we will obtain an integral formula for elements of the Soblev spaces with non-negative smoothness. Of course, for sufficiently smooth sections such formulae are well known (see, for instance, [15, §2.4]).

From now on we additionally assume that the operators Δ_i, $0 \leq i \leq N$, satisfy the Uniqueness Condition 1.1. Then each of these operators has a bilateral pseudo-differential fundamental solution, say, Φ_i, on $\overset{\circ}{X}$ (see, for example, [2, §4.4.2]). Schwartz kernel of the operator Φ_i is denoted by $\Phi_i(x, y)$, $x \neq y$. It is known, that $\Phi_i(x, y) \in C^\infty((E_i \otimes E^*_i) \setminus \{x = y\})$ (see, for instance, [15, §5]).

For a section $f \in C^\infty(\overline{D}, E_{i+1})$ we denote by $T_i f$ the following volume potential:

$$T_i f(x) = (\Phi_i A^*_i \chi_D f)(x) = \int_D \langle (A^*_i)^T_y \Phi_i(x, \cdot), f \rangle_y dy.$$

If ∂D is smooth enough (e.g. $\partial D \in C^\infty$) then the potential T_i induces a bounded linear operator

$$T_i : H^{s-1}(D, E_{i+1}) \to H^s(D, E_i), \qquad s \in \mathbb{N}$$

(see, for example, [31, 1.2.3.5]).

Lemma 3.1. *For any domain $\Omega \Subset \overset{\circ}{X}$ with $\partial\Omega \in C^\infty$ the potential T_i induces a bounded linear operator*

$$T_{i,\Omega} : H^{-s}(D, E_{i+1}) \to H^{-s+1}_{A_i}(\Omega, E_i), \quad s \in \mathbb{N}.$$

Moreover for every section $f \in H^{-s}(D, E_{i+1})$ it is true that $\Delta_i T_{i,\Omega} f = A^*_i \chi_D f$ in $\Omega \setminus \overline{D}$.

Proof. First of all we note that any smoothing operator $\tilde{K}$ of type $E_{i+1} \to E_i$ on $\overset{\circ}{X}$ induces for any p a bounded linear operator

$$\tilde{K}\chi_D : H^{-s}(D, E_{i+1}) \to C^p(\overline{\Omega}, E_i).$$

As any two fundamental solutions differ on a smoothing operator, we may assume that $\Phi_i = \mathcal{G}^{(X)}_{\Delta_i}$. The principal advantage of $\mathcal{G}^{(X)}_{\Delta_i}$ is in the fact that the volume potential is $L^2(X, E_i)$-self-adjoint (see, for instance, [28, formula (2.75)]). Besides, it has the transmission property (see [31, §2.2.2]) and hence it continuously acts on the Sobolev scale:

$$\mathcal{G}^{(X)}_{\Delta_i}\chi_D : H^{s-1}(D, E_i) \to H^{s+1}(\Omega, E_i), \quad \mathcal{G}^{(X)}_{\Delta_i}A_i^*\chi_D : H^{s-1}(D, E_{i+1}) \to H^{s}(\Omega, E_i), \quad s \in \mathbb{N}.$$

In particular, $\mathcal{G}^{(X)}_{\Delta_i}\chi_\Omega v$ belongs to $H^2_{loc}(\overset{\circ}{X}, E_i) \cap C^\infty(\overline{\Omega}, E_i)$ for all $v \in C^\infty(\overline{\Omega}, E_i)$ and, similarly, $\mathcal{G}^{(X)}_{\Delta_i}A_i^*\chi_\Omega g$ belongs to $H^1_{loc}(\overset{\circ}{X}, E_i) \cap C^\infty(\overline{\Omega}, E_i)$ for all $g \in C^\infty(\overline{\Omega}, E_{i+1})$. Then for all $f \in C^\infty(\overline{D}, E_{i+1})$, $v \in C^\infty(\overline{\Omega}, E_i)$, $g \in C^\infty(\overline{\Omega}, E_{i+})$ we have:

$$(T_i f, v)_\Omega = (\mathcal{G}^{(X)}_{\Delta_i}A_i^*\chi_D f, \chi_\Omega v)_X = (\chi_D f, A_i\mathcal{G}^{(X)}_{\Delta_i}\chi_\Omega v)_X,$$

$$(A_iT_i f, g)_\Omega = (A_i\mathcal{G}^{(X)}_{\Delta_i}A_i^*\chi_D f, \chi_\Omega g)_X = (\chi_D f, A_i\mathcal{G}^{(X)}_{\Delta_i}A_i^*\chi_\Omega g)_X.$$

Therefore, we have

$$\|T_i f\|_{-s,A_i,\Omega} \le C_1\,\|f\|_{-s-1,D} \text{ for all } f \in C^\infty(\overline{D}, E_{i+1}), \tag{11}$$

$$\|A_iT_i f\|_{-s-1,A_i,\Omega} \le C_2\,\|f\|_{-s-1,D} \text{ for all } f \in C^\infty(\overline{D}, E_{i+1}), \tag{12}$$

with positive constants C_1, C_2 do not depending on f.

Let now $f \in H^{-s-1}(D, E^{i+1})$. Then there is a sequence $\{f_\nu\} \subset C^\infty(\overline{D}, E_{i+1})$ converging to f in $H^{-s-1}(D, E_{i+1})$. According to (11), (12) the sequence $\{T_i f_\nu\}$ is fundamental in the space $H^{-s}_{A_i}(\Omega, E_i)$; its limit we denote $T_{i,\Omega}f$. It is easy to understand that this limit does not depend on the choice of the sequence $\{f_\nu\}$ converging to f, and the estimates (11), (12) guarantee that the operator $T_{i,\Omega}$, defined in this way, is bounded. Moreover, the properties of the fundamental solutions Φ_i means that each of the potentials $T_i f_\nu$ satisfies

$$(T_i f_\nu, \Delta_i v)_\Omega = 2\langle A_i^*\chi_D f_\nu, v\rangle = (\chi_D f_\nu, A_i v)_\Omega \text{ for all } v \in C^\infty_{comp}(\Omega \setminus \overline{D}, E_i).$$

Passing to the limit with respect to $\nu \to \infty$ in the last equality we obtain the desired statement because the operators χ_D and $T_{i,\Omega}$ are continuous. $\square$

Further, for a section $v \in C^\infty(\overline{D}, E_i)$ we denote by $K_i f$ the following volume potential:

$$K_i v = (\Phi_i A_{i-1} - A_{i-1}\Phi_{i-1})A^*_{i-1}\chi_D v.$$

Again, by the definition, it is a zero order pseudo-differential operator with the transmission property. If ∂D is smooth enough (e.g. $\partial D \in C^\infty$) then the potential K_i induces a bounded linear operator

$$K_i : H^s(D, E_i) \to H^s(D, E_i), \qquad s \in \mathbb{Z}_+$$

(see, for example, [31, 1.2.3.5]).

Lemma 3.2. *For any domain $\Omega \Subset \overset{\circ}{X}$ with $\partial\Omega \in C^\infty$ the operator K_i induces a smoothing operator on $\overline{\Omega}$. In particular, for all $s \in \mathbb{N}$, $p \in \mathbb{N}$, it is bounded linear operator*

$$K_{i,\Omega} : H^{-s}(D, E_i) \to C^p(\overline{\Omega}, E_i) \cap S_{\Delta_i}(\Omega).$$

Proof. Indeed, by the definition of the fundamental solution,

$$\Delta_i(\Phi_i A_{i-1} - A_{i-1}\Phi_{i-1})v = A_{i-1}v - A_{i-1}v = 0 \text{ for all } v \in C^\infty_{comp}(\overset{\circ}{X}, E_{i-1}).$$

Therefore the pseudo-differential operator $(\Phi_i A_{i-1} - A_{i-1}\Phi_{i-1})$ (of order (-1) on X) is smoothing on compact subsets of $\overset{\circ}{X}$. Now the similar statements follows for K_i. $\square$

For $x \notin \partial D$ we denote $M_i v_0(x)$ the following Green integral with a density $v_0 \in C^\infty(\partial D, E_i)$:

$$M_i v_0(x) = -\int_{\partial D} G_{A_i}(\star A_i \star^{-1} \Phi_i(x,\cdot), v_0) = -\int_{\partial D} (\tau_i(v_0), \tilde{\nu}_{i+1}(A_i \star^{-1} \Phi_i(x,\cdot))_y \, ds(y), \ x \notin \partial D; \tag{13}$$

the last identity easily follows from (8). Thus we define the Green transform with a density $v_0 \in \mathcal{D}'(\partial D, E_i)$ as the result of the action of the distribution v_0 on the "test-function" $(-\tilde{\nu}_i(A_i \star^{-1} \Phi_i(x,\cdot)) \in C^\infty(\partial D, E_i)$:

$$M_i v_0(x) = -(v_0, \tilde{\nu}_{i+1}(A_i \star^{-1} \Phi_i(x,\cdot))_{\partial D} = -(\tau_i(v_0), \tilde{\nu}_{i+1}(A_i \star^{-1} \Phi_i(x,\cdot))_{\partial D}, \qquad x \notin \partial D.$$

By the construction, $M_i v_0 \in S_{\Delta_i}(\overset{\circ}{X} \setminus \text{supp } v_0, E_i)$ as a parameter dependent distribution; here supp v_0 is the support of v_0.

Again, if ∂D is smooth enough (e.g. $\partial D \in C^\infty$) then the potential M_i induces a bounded linear operator

$$M_i : H^{s-1/2}(\partial D, E_i) \to H^s(D, E_i), \qquad s \in \mathbb{N}$$

(see, for example, [31, 1.2.3.5]).

Now using Stokes formula and the potentials T_i, M_i, K_i we arrive to a homotopy formula for the complex $\{A_i\}$ and sections $u \in C^\infty(\overline{D}, E_i)$ (see [15, Theorem 2.4.8]):

$$M_i u + T_i A_i u + A_{i-1} T_{i-1} u + K_i u = \chi_D u. \tag{14}$$

Of course, the continuity of the operators T_i, M_i, K_i on the Sobolev spaces implies that formula (14) is still valid for all the sections $u \in H^s(D, E_i)$, $s \in \mathbb{N}$. We are to extend the homotopy formula for the complex $\{A_i\}$ on the scale $H^{-s}_{A_i}(D, E_i)$, $s \in \mathbb{Z}_+$.

Lemma 3.3. *For any domain $\Omega \Subset \overset{\circ}{X}$ such that $\partial\Omega \in C^\infty$ and $D \subset \Omega$ the potential M induces bounded linear operators*

$$M_{i,D} : H^{-s-1/2}(\partial D, E_i) \to H^{-s}_{A_i}(D, E_i), \quad M_{i,\Omega} : H^{-s-1/2}(\partial D, E_i) \to H^{-s}(\Omega, E_i).$$

Proof. As we already have seen above (see Remark 2.1 and Corollary 2.3), for every section $v^0 \in H^{-s-1/2}(\partial D, E_i)$ the Poisson integral $\mathcal{P}^{(D)}_{\Delta_i} v^0 \in H^{-s}_{A_i \oplus A^*_{i-1}}(D, E_i)$ satisfies $t_i(\mathcal{P}^{(D)}_{\Delta_i} v^0) = v^0$. Set

$$M_{i,D} = (I - T_{i,D} A_i - A_{i-1} T_{i,D} - K_{i,D}) \, \mathcal{P}^{(D)}_{\Delta_i} : H^{-s-1/2}(\partial D, E_i) \to H^{-s}_{A_i}(D, E_i),$$

$$M_{i,\Omega} = (\chi_D - T_{i,\Omega}A_i - A_{i-1}T_{i,\Omega} - K_{i,\Omega})\,\mathcal{P}^{(D)}_{\Delta_i} : H^{-s-1/2}(\partial D, E_i) \to H^{-s}(\Omega, E_i).$$

It follows from Lemmas 3.1, 3.2 and the continuity of the operators $\mathcal{P}^{(D)}_{\Delta_i}$ and χ_D that the defined above operators $M_{i,D}$, $M_{i,\Omega}$ are bounded. Let us see that $M_{i,D}$ and $M_{i,\Omega}$ coincide with M_i on $C^\infty(\partial D, E_i)$. Indeed, if $v^0 \in C^\infty(\partial D, E_i)$ then Remark 2.1 implies $\mathcal{P}^{(D)}_{\Delta i} v^0 \in C^\infty(\overline{D}, E_i)$ and

$$M_i v^0 = M_i \mathcal{P}^{(D)}_{\Delta_i} v^0 = M_i \tau_i(\mathcal{P}^{(D)}_{\Delta_i} v^0).$$

Now using a homotopy formula (14) we obtain:

$$\chi_D \mathcal{P}^{(D,i)}_{\Delta_i} v^0 = M_i v^0 + T_{i,D} A_i \mathcal{P}^{(D)}_{\Delta_i} v^0 + A_{i-1} T_{i-1,D} \mathcal{P}^{(D)}_{\Delta_i} v^0 + K_i \mathcal{P}^{(D)}_{\Delta_i} v^0.$$

Since $C^\infty(\partial D, E_i)$ is dense in $H^{-s-1/2}(\partial D, E_i)$ then M_i continuously extends from $C^\infty(\partial D, E_i)$ onto $H^{-s-1/2}(\partial D, E_i)$ as defined above operators $M_{i,D}$, $M_{i,\Omega}$. Moreover, it is easy to understand that the sections $M_{i,D}v^0$, $M_{i,\Omega}v^0$ are coincide with the distributions Mv^0 on D and $\Omega \setminus \operatorname{supp} v^0$ respectively. $\square$

Theorem 3.1. *For every section $u \in H_{A_i}(D, E_i)$ the following formulae hold:*

$$M_{i,D} u + T_{i,D} A_i u + A_{i-1} T_{i-1,D} u + K_{i,D} u = u, \tag{15}$$

$$M_{i,\Omega} u + T_{i,\Omega} A_i u + A_{i-1} T_{i-1,\Omega} u + K_{i,\Omega} u = \chi_D u. \tag{16}$$

Proof. Pick $u \in H_{A_i}(D, E_i)$. Then $u \in H^{-s}_{A_i}(D, E_i)$ with a number $s \in \mathbb{Z}_+$ and there is $\{u_\nu\} \subset C^\infty(\overline{D}, E_i)$ converging to u in the space $H^{-s}_{A_i}(D, E_i)$. Now the homotopy formula (14) implies

$$M_i u_\nu + T_i A_i u_\nu + A_{i-1} T_i u_\nu + K_i u_\nu = \chi_D u_\nu. \tag{17}$$

Passing to the limit in the spaces $H^{-s}_{A_i}(D, E_i)$ and $H^{-s}(\Omega, E_i)$ with respect to $\nu \to \infty$ in (17) we obtain (15) and (16) respectively because of Lemmas 3.1, 3.2, 3.3. $\square$

Remark 3.1. Let $f \in H^{-s-1}(D, E_{i+1})$. If Ω, Ω_1 are bounded domains in $\overset{\circ}{X}$ (with smooth boundaries) containing D then sections $T_{i,\Omega} f \in H^{-s}(\Omega, E_i)$ and $T_{i,\Omega_1} f \in H^{-s}(\Omega_1, E_i)$ belong to $S_{\Delta_i}(\Omega \setminus \overline{D})$ and $S_{\Delta_i}(\Omega_1 \setminus \overline{D})$ respectively. Since they are constructed as the limits of the same sequence of sections converging in different spaces, they coincide in $(\Omega_1 \cap \Omega) \setminus \overline{D}$. The same conclusion is obviously valid for the smoothing operators $K_{i,\Omega}$ and K_{i,Ω_1}. Moreover, as the operators $M_{i,\Omega}$ and M_{i,Ω_1} are constructed with the use of $T_{i,\Omega}$, $K_{i,\Omega}$ and T_{i,Ω_1}, K_{i,Ω_1} respectively, this is also true for the sections of the type $M_{i,\Omega}(v^0)$ with $v^0 \in H^{s-1/2}(\partial D, E_i)$. Since $\Omega \subset \overset{\circ}{X}$ is arbitrary, the Uniqueness Condition 1.1 allows us to say about the sections $T_i f$ and $M_i v^0$ from $S^F_{\Delta_i}(\overset{\circ}{X} \setminus \overline{D})$ such that $T_i f = T_{i,\Omega} f \in H^{-s}(\Omega, E_i)$, $M_i v^0 = M_{i,\Omega} v^0 \in H^{-s}(\Omega, E_i)$ for any domain $\Omega \supset D$. $\square$

4 The Cauchy problem in spaces of distributions

Problem 4.1. *Given $u_0 \in \mathcal{D}'(\overline{\Gamma}, E_i)$, $f \in H_{A_{i+1}}(D, E_{i+1})$ find a section $u \in H_{A_i}(D, E_i)$ such that*

$$A_i u = f \text{ in } D, \qquad \tau_i(u) = \tau_i(u_0) \text{ on } \Gamma,$$

in the sense of Definition 2.1, i.e.

$$(u, A_i^* g)_D = (f, g)_D - \langle \star \tilde{\nu}_{i+1}(g), \tau_i(u_0) \rangle_\Gamma \text{ for all } g \in C^\infty_{comp}(D \cup \Gamma, E_{i+1}). \tag{18}$$

If $i = 0$ then A_0 has the injective principal symbol and the Cauchy problem has no more than one solution (see, for instance, [2, Theorem 10.3.5]). Clearly it may have infinitely many solutions if $i > 0$. Usually the Uniqueness Theorem of the Cauchy problem for $i > 0$ is valid in co-homologies under some convexity conditions on $\partial D \setminus \Gamma$ (cf. [18, Corollary 3.2]). Instead of looking for a version of Uniqueness Theorem we will try to choose a canonic solution of the Cauchy problem (see §5 below for solutions in $H^0_{A_i}(D, E_i)$).

We easily see that f and u^0 should be coherent. Namely, as $A_i^* A_{i+1}^* \equiv 0$, taking $g = A_{i+1}^* w$ with $w \in C^\infty_{comp}(D \cup \Gamma, E_{i+2})$ in (18) we conclude that for the solvability of problem 4.1 it is necessary that

$$(f, A_{i+1}^* w)_D = \langle \star \tilde{\nu}_{i+1}(A_{i+1}^* w), \tau_i(u_0) \rangle_\Gamma \text{ for all } w \in C^\infty_{comp}(D \cup \Gamma, E_{i+2}). \tag{19}$$

Let us discuss this. First we note that, due to Corollary 2.6 and to the properties of the complex, $A_{i+1} f = 0$ in D if the Cauchy problem is solvable. This corresponds to $w \in C^\infty_{comp}(D, E_{i+2})$ in (19).

Besides, the operator A_i induces the tangential operator $\{A_{i,\tau}\}$ on ∂D (see, for instance, [15, §3.1.5]). More precisely, let $\hat{u}^0 \in \mathcal{D}'(\partial D, E_i)$. Pick a section $\hat{u} \in H_{A_i}(D, E_i)$ satisfying $\tau_i(\hat{u}) = \tau_i(\hat{u}^0)$ on ∂D (there is at least one such a section, $\mathcal{P}^{(D)}_{\Delta_i} \tau_i(\hat{u}^0)$). Then set $A_{i,\tau} \hat{u}^0 = \tau_{i+1}(A_i \hat{u})$. If we fix $g \in C^\infty(\partial D, E_{i+1})$ then, by Remark 2.1, the section $w = \mathcal{P}^{(D)}_{\Delta_{i+2}} \tilde{\tau}_{i+1}(g)$ belongs to the space $C^\infty(\overline{D}, E_{i+2})$. Now, easily, Definition 2.1 and Lemma 2.1 imply that

$$\begin{aligned} \langle \star g, A_{i,\tau} \hat{u}^0 \rangle &= \langle \star \tilde{\nu}_{i+2}(\tilde{\tau}_{i+1}(g)), \tau_{i+1}(A_i \hat{u}) \rangle = \langle \star \tilde{\nu}_{i+2}(w), \tau_{i+1}(A_i \hat{u}) \rangle = \\ &(A_i \hat{u}, A_{i+1}^* w)_D = \langle \star \tilde{\nu}_{i+1}(A_{i+1}^* w), \tau_i(\hat{u}) \rangle = \langle \star \tilde{\nu}_{i+1}(A_{i+1}^* w), \tau_i(\hat{u}^0) \rangle. \end{aligned} \tag{20}$$

In particular, this means that $A_{i,\tau} \hat{u}^0$ does not depend on the choice of $\hat{u} \in H_{A_i}(D, E_i)$ with $\tau_i(\hat{u}) = \tau_i(\hat{u}^0)$ on ∂D.

Lemma 4.1. *For the Cauchy data u_0 and f, identity* (19) *holds if and only if $A_{i+1} f = 0$ in D and $\tau_{i+1,\Gamma}(f) = A_{i,\tau} u^0$ on Γ.*

Proof. Indeed, as we have noted above, (19) implies $A_{i+1} f = 0$ in D. Then, similarly to (20), it follows from Definition 2.1 that, with $w = \mathcal{P}^{(D)}_{\Delta_{i+2}} \tilde{\tau}_{i+1}(g)$,

$$\langle \star g, \tau_{i+1}(f) \rangle = \langle \star \tilde{\nu}_{i+2}(\tilde{\tau}_{i+1}(g)), \tau_{i+1}(f) \rangle = \langle \star \tilde{\nu}_{i+2}(w), \tau_{i+1}(f) \rangle = (f, A_{i+1}^* w)_D$$

for all $g \in C^\infty(\partial D, E_{i+1})$ if $A_{i+1} f = 0$ in D. Therefore taking $\hat{u}^0 = u^0$ on Γ and $g \in C^\infty_{comp}(\Gamma, E_{i+1})$ in (20) we conclude that $\tau_{i+1,\Gamma}(f) = A_{i,\tau} u^0$ on Γ too, if identity (19) holds.

Back, if $A_{i+1} f = 0$ in D and $\tau_{i+1,\Gamma}(f) = A_{i,\tau} u^0$ on Γ then , again applying Definition 2.1 and calculating as in (20), we obtain for all $w \in C^\infty_{comp}(D \cup \Gamma, E_{i+2})$:

$$(f, A_{i+1}^* w)_D = \langle \star \tilde{\nu}_{i+2}(w), \tau_{i+1}(f) \rangle = \langle \star \tilde{\nu}_{i+2}(w), A_{i,\tau} u^0 \rangle = \langle \star \tilde{\nu}_{i+1}(A_{i+1}^* w), \tau_i(u^0) \rangle,$$

which was to be proved. $\square$

It is important to note that Lemma 4.1 allows the point wise check of necessary solvability conditions for Problem 4.1, at least if the Cauchy data f and u^0 are smooth.

Now choose a domain D^+ in such a way that the set $\Omega = D \cup \Gamma \cup D^+$ is a bounded domain with smooth boundary in $\overset{\circ}{X}$. It is convenient to denote $F^{\pm}$ the restrictions of a section F onto $D^{\pm}$ (here $D^- = D$).

Further, for $u^0 \in H^{-s-1/2}(\overline{\Gamma}, E_i)$, choose a representative $\tilde{u}^0 \in H^{-s-1/2}(\partial D, E_i)$. We have seen above the potentials $M_i \tilde{u}_0$ and $T_i f$ satisfy $\Delta_i (M_i \tilde{u}_0) = 0$ and $\Delta_i (T_i f) = 0$ everywhere outside $\overline{D}$ as parameter dependent distributions. Hence the section

$$F_i = M_{i,\Omega} \tau_i (\tilde{u}_0) + T_{i,\Omega} f$$

belongs to $S_{\Delta_i}(D^+) \cap H(\Omega, E_i)$. The Green formula (16) shows that the potential F_i contains a lot of information on solvability conditions of Problem 4.1.

Denote $\chi_D(H(D, E_i))$ the image of the space $H(D, E_i)$ under the map $\chi_D : H(D, E_i) \to H(\Omega, E_i)$ (see map (5)).

Theorem 4.1. *Let Δ_{i-1}, Δ_i, Δ_{i+1} satisfy the Uniqueness Condition* 1.1. *Then the Cauchy Problem* 4.1 *is solvable if and only if condition* (19) *holds true and there is a section $\mathcal{F}_i \in H(\Omega, E_i)$ such that $A_i \Delta_i \mathcal{F}_i = 0$ in Ω and $(F_i - \mathcal{F}_i) \in \chi_D(H(D, E_i))$.*

Proof. Let Problem 4.1 be solvable and u be its solution. The necessity of condition (19) is already proved. Set

$$\mathcal{F}_{i,u} = M_{i,\Omega} \tau_i (\tilde{u}^0) + T_{i,\Omega} f - \chi_D u. \tag{21}$$

Lemmas 3.1, 3.1, 3.3 and Remark 3.1 imply that $\mathcal{F}_{i,u} \in H^{-s}(\Omega, E_i)$ with some $s \in \mathbb{Z}_+$. Clearly $(F_i - \mathcal{F}_i) = \chi_D u \in \chi_D(H(D, E_i))$. Then it follows from homotopy formula (16) that:

$$\mathcal{F}_{i,u} = M_{i,\Omega}(\tau_i(\tilde{u}^0) - \tau_i(u))) - A_{i-1} T_{i-1,\Omega} u - K_i u. \tag{22}$$

Since $(\tau_i(\tilde{u}^0) - \tau_i(u)) = 0$ on Γ then $M_{i,\Omega}(\tau_i(\tilde{u}^0) - \tau_i(u))$ belongs to $S_{\Delta_i}(\overset{\circ}{X} \setminus \Gamma)$ as a parameter dependent distribution. That is why, using Lemma 3.1, we obtain:

$$\Delta_i \mathcal{F}_{i,u} = -\Delta_i A_{i-1} T_{i-1,\Omega} u = -A_{i-1} \Delta_i T_{i-1,\Omega} u = -A_{i-1} A^*_{i-1} \chi_D u \text{ in } \Omega. \tag{23}$$

In particular, $A_i \Delta_i \mathcal{F}_{i,u} = 0$ in Ω.

Back, let there be sections $\mathcal{F}_i \in H(\Omega, E_i)$ and $u \in H(D, E_i)$ such that $A_i \Delta_i \mathcal{F}_i = 0$ in Ω and

$$\chi_D u = F_i - \mathcal{F}_i. \tag{24}$$

Let us show that the section u is a solution to Problem 4.1. With this aim we consider the following functional $w(\tilde{u}^0)$ on the space $C^\infty(\overline{D}, E_{i+1})$:

$$\langle w(\tilde{u}^0), v \rangle = (\tau_i(\tilde{u}^0), \tilde{\nu}_{i+1}(v))_{\partial D} \text{ for all } v \in C^\infty(\overline{D}, E_{i+1}).$$

As $\tilde{u}^0 \in \mathcal{D}'(\partial D, E_i)$ then $\tilde{u}^0 \in H^{-s-1/2}(\partial D, E_i)$ with some $s \in \mathbb{Z}_+$ and hence for all $v \in C^\infty(\overline{D}, E_{i+1})$ we have:

$$|\langle w(\tilde{u}^0), v \rangle| \le \|\tau_i(\tilde{u}^0)\|_{-s-1/2,\partial D} \|\tilde{\nu}_{i+1}(v)\|_{s+1/2,\partial D} \le C\ \|\tau_i(\tilde{u}^0)\|_{-s-1/2,\partial D} \|v\|_{s+1,D}$$

with a constant $C > 0$ which does not depend on $\tilde{u}^0$ and v. Therefore $w(\tilde{u}^0) \in H^{-s-1}(D, E_{i+1})$ and its support belongs to ∂D.

Clearly, $C^\infty_{comp}(D \cup \Gamma, E_{i+1}) \subset C^\infty_{comp}(\Omega, E_{i+1})$ and Whitney theorem implies that every section from $C^\infty_{comp}(D \cup \Gamma, E_{i+1})$ may be extended up to an element of the space $C^\infty_{comp}(\Omega, E_{i+1})$. Thus, (18) is equivalent to the following identity:

$$g = A_i \chi_D u - \chi_D f + \chi_D w(\tilde{u}^0) \equiv 0 \text{ in } \Omega. \tag{25}$$

That is why u is a solution to Problem 4.1 if and only if $u \in H_{A_i}(D, E_i)$ and the identity (25) holds. By the very construction, g belongs to $\mathcal{D}'(\Omega, E_{i+1})$ and its support lies in $\overline{D}$.

Then for all $v \in C^\infty_{comp}(\Omega, E_{i+1})$ we have

$$\langle g, \Delta_{i+1} v\rangle_\Omega = (\chi_D u, A_i^* \Delta_{i+1} v)_\Omega - (\chi_D f, \Delta_{i+1} v)_\Omega + (\chi_D w(\tilde{u}^0), \Delta_{i+1} v)_\Omega$$

$$(F_i - \mathcal{F}_i, \Delta_i A_i^* v)_\Omega - (f, \Delta_{i+1} v)_D + (\tau_i(\tilde{u}^0), \tilde{\nu}_{i+1}(\Delta_{i+1} v))_{\partial D} =$$
$$(F_i, \Delta_i A_i^* v)_\Omega - (f, \Delta_{i+1} v)_D + (\tau_i(\tilde{u}^0), \tilde{\nu}_{i+1}(\Delta_{i+1} v))_{\partial D}, \tag{26}$$

because $A_i^* \Delta_{i+1} = \Delta_i A_i^*$ and $A_i \Delta_i \mathcal{F}_i = 0$ in Ω.

Further, by Lemma 3.1, we see that for all $v \in C^\infty_{comp}(\Omega, E_{i+1})$,

$$(T_{i,\Omega} f, \Delta_i A_i^* v)_\Omega = (A_i^* \chi_D f, A_i^* v)_\Omega = (f, A_i A_i^* v)_D. \tag{27}$$

Set $\tilde{u} = \mathcal{P}^{(D)}_{\Delta_i} \tau_i(\tilde{u}^0)$. This section belongs to $H_{A_i \oplus A^*_{i-1}}(D, E_i)$ (see Remark 2.1 and Corollary 2.3). By the definition, $\tau_i(\tilde{u}) = \tau_i(\tilde{u}_0)$ on ∂D. Now Lemma 3.3, the properties of the fundamental solutions and Definition 2.1 imply that for all $v \in C^\infty_{comp}(\Omega, E_{i+1})$ we have:

$$(M_{i,\Omega} \tau_i(\tilde{u}^0), \Delta_i A_i^* v)_\Omega = (\chi_D \tilde{u} - T_{i,\Omega} A_i u - A_{i-1} T_{i-1,\Omega} \tilde{u} - K_{i,\Omega} u, \Delta_i A_i^* v)_\Omega =$$

$$(\tilde{u}, A_i^* A_i A_i^* v)_D - (A_i \tilde{u}, A_i A_i^* v)_D = -(\tau_i(\tilde{u}^0), \tilde{\nu}_{i+1}(A_i A_i^* v)_{\partial D}. \tag{28}$$

Therefore, using (26), (27), (28) we conclude that

$$\langle g, \Delta_{i+1} v\rangle_\Omega = -(f, A^*_{i+1} A_{i+1} v)_D + (\tau_i(\tilde{u}^0), \tilde{\nu}_{i+1}(A^*_{i+1} A_{i+1} v))_{\partial D} = 0$$

for all $v \in C^\infty_{comp}(\Omega, E_{i+1})$ because of condition (19).

Thus, $\Delta_{i+1} g = 0$ in Ω and $g = 0$ in D^+. It follows from Uniqueness Condition 1.1 that $g \equiv 0$ in Ω, i.e. identity (18) holds. In particular this means that $A_i u = f$ in D and, by Corollary 2.5, we see that $u \in H_{A_i}(D, E_i)$, which was to be proved. □

Corollary 4.1. *Let $f \in H^{-s-1}(D, E_{i+1})$, $u^0 \in H^{-s-1/2}(\overline{\Gamma}, E_i)$. The Cauchy problem* 4.1 *is solvable in the space $H^{-s}_{A_i}(D, E_i)$ if and only if condition* (19) *is fulfilled and there is a section $\mathcal{F}_i \in H^{-s}(\Omega, E_i)$ satisfying $A_i \Delta_i \mathcal{F}_i = 0$ in Ω and such that $(F_i - \mathcal{F}_i) \in \chi_D(H^{-s}(D, E_i))$.*

Proof. Indeed, if Problem 4.1 is solvable in $H^{-s}_{A_i}(D, E_i)$, then condition (19) is fulfilled and $\mathcal{F}_i = F_i - \chi_D u$ (see (21)). Hence, by Lemma 3.3, the section $\mathcal{F}$ belongs to $H^{-s}(\Omega, E_i)$ and $(F_i - \mathcal{F}_i) \in \chi_D(H^{-s}(D, E_i))$.

Back, if condition (19) is fulfilled, $\mathcal{F}_i \in H^{-s}(\Omega, E_i)$ satisfies $A_i \Delta_i \mathcal{F}_i = 0$ in Ω and $(F_i - \mathcal{F}_i) \in \chi_D(H^{-s}(D, E_i))$ then Problem 4.1 is solvable. Besides, one of its solutions u is given by formula

(24). In particular, $\chi_D u = (F_i - \mathcal{F}_i)$ belongs to $H^{-s}(\Omega, E_i)$. Pick $v \in C^\infty(\overline{D}, E_i)$. Then, by Whitney Theorem, there is a section $V \in C^\infty(\overline{\Omega}, E_i)$ with $\|V\|_{s,\Omega} = \|v\|_{s,D}$ and $v = V$ in D. By the definition,

$$|(u, v)_D| = |(\chi_D u, V)_\Omega| \leq \|\chi_D u\|_{-s,\Omega} \|v\|_{s,D},$$

i.e. $u \in H^{-s}(D, E_i)$. Finally, as $A_i u = f \in H^{-s-1}(D, E_{i+1})$, then $u \in H^{-s}_{A_i}(D, E_i)$ according to Corollary 2.5. $\square$

If $i = 0$ then the operator A_0 has injective principal symbol and Theorem 4.1 has the following form (cf. [2], [12] for the operators with real analytic coefficients and $f = 0$).

Corollary 4.2. *Let $f \in H(D, E_1)$, $u^0 \in \mathcal{D}'(\overline{\Gamma}, E_0)$. The Cauchy Problem* 4.1 *is solvable in the space $H_{A_0}(D, E_0)$ if and only if condition* (19) *is fulfilled and there is a section $\mathcal{F}_0 \in H(\Omega, E_0)$, coinciding with F_0 in D^+ and such that $\Delta_0 \mathcal{F}_0 = 0$ in Ω.*

Proof. If $i = 0$ then the operator A^*_{-1} in (23) equals to zero and therefore $\Delta_0 \mathcal{F}_0 = 0$ in Ω.

Back, as $\Delta_0 \mathcal{F}_0 = 0$ then the section $\mathcal{F}_0$ is smooth in Ω. According to [2, Theorem 9.4.8] the section $\mathcal{F}_0$ belongs to $H(\Omega, E_0)$ if and only if it has finite order of growth near $\partial\Omega$. As $D \subset \Omega$, the section $\mathcal{F}_0^-$ has the same order of growth (in D) near ∂D. Then $\mathcal{F}_0^- \in H(D, E_0)$, $u = F_0^- - \mathcal{F}_0^-$ in $H(D, E_0)$ and $(F_0 - \mathcal{F}_0) \in \chi_D(H(D, E_0))$ because $F_0 = \mathcal{F}_0$ in D^+. $\square$

In the next section we will obtain a similar result in positive degrees of the complex $\{A_i\}$ over Lebesgue space $L^2(D, E_i)$ choosing a canonical solution u in (22). In any case, Theorem 4.1 can be easily reformulated to be like Corollary 4.2

Corollary 4.3. *The Cauchy Problem* 4.1 *is solvable if and only if condition* (19) *is fulfilled and there is a section $\mathcal{F}_i \in H(\Omega, E_i)$ such that $(F_i - \mathcal{F}_i) \in \chi_D(H(D, E_i))$ and $\Delta_i \mathcal{F}_i$ co-homological to zero in Ω with respect to the complex $\{A_i\}$.*

Proof. It follows from Theorem 4.1 and (23) because $A_i \circ A_{i-1} \equiv 0$. $\square$

5 The Cauchy problem in the Lebesgue space

Consider now the case $s = 0$. Denote Σ_0 the null space of the Cauchy Problem 4.1 for $s = 0$, i.e. Σ_0 consists of $L^2(D, E_i)$-sections w with $A_i w = 0$ in D and $\tau_i(w) = 0$ on Γ, or, the same

$$(w, A_i^* v)_D = 0 \text{ for all } v \in C^\infty_{comp}(D \cup \Gamma, E_{i+1}). \tag{29}$$

Formula (29) guarantees that Σ_0 is a (closed) subspace in $L^2(D, E_i)$.

As the adjoint complex $\{A_i^*\}$ is elliptic too we may give similar definition of weak boundary value of a normal part (with respect to $\{A_i\}$) of a section on Γ.

Definition 5.1. *We say that a section $u \in H_{A^*_{i-1}}(D, E_i)$, satisfying $A^*_{i-1} u = h$ in D with $h \in H(D, E_{i-1})$, has a weak boundary value $\nu_{i,\Gamma}(u) = \nu_i(u_0)$ on Γ for $u_0 \in \mathcal{D}'(\Gamma, E_i)$ if*

$$(h, g)_D - (u, A_{i-1} g)_D = \langle \star\tilde{\tau}_{i-1}(g), \nu_i(u_0) \rangle_\Gamma \text{ for all } g \in C^\infty_{comp}(D \cup \Gamma, E_{i-1}).$$

Theorem 5.1. *Let $f \in H^{-1}(D, E_{i+1})$, $u^0 = 0$. If the Cauchy Problem* 4.1 *is solvable in $H^0_{A_i}(D, E_i)$ then its unique $L^2(D, E_i)$ -orthogonal to Σ_0 solution $u(f)$ satisfies $\nu_{i,\Gamma}(u(f)) = 0$ on Γ in the sense of Definition 5.1 and $A^*_{i-1} u(f) = 0$ in D.*

Proof. Obviously, $H^0_{A_i}(D,E_i)$ is a Hilbert space with the scalar product

$$(\cdot,\cdot)_{0,A_i} = (\cdot,\cdot)_0 + (A_i\cdot, A_i\cdot)_{-1}.$$

Then the orthogonal complement to Σ_0 in this space coincides with $L^2(D,E_i)$-orthogonal complement to Σ_0. Thus, if the Cauchy Problem 4.1 has a solution u in $H^0_{A_i}(D,E_i)$ then its $L^2(D,E_i)$ -orthogonal projection $u(f)$ to the orthogonal complement to Σ_0 is also a solution to Problem 4.1 (it is evidently unique with the prescribed property). Clearly, any section of the type $A_{i-1}\phi$, with $v \in C^\infty_{comp}(D,E_{i-1})$, belongs to Σ_0. Hence

$$(u(f), A_{i-1}v)_D = 0 \text{ for all } v \in C^\infty_{comp}(D,E_{i-1}),$$

and then $A^*_{i-1}u(f) = 0$ in D.

Now, according to Corollaries 2.2 and 2.7, the section $u(f)$ has traces of $\nu_i(u(f))$ on ∂D, belonging to $H^{-1/2}(D,E_i)$. Hence, by Definition 5.1, the normal part $\nu_{i,\Gamma}(u)f))$ vanishes on Γ if and only if

$$(u(f), A_{i-1}v)_D = 0 \text{ for all } v \in C^\infty_{comp}(D\cup\Gamma,E_{i-1}). \tag{30}$$

Further, it follows from Corollary 2.2 that the space $H^0_{A_i}(D,E_i)$ is the Hilbert space with the scalar product

$$(\cdot,\cdot)_{0,\tau_i} = (\cdot,\cdot)_0 + (\tau_i\cdot, \tau_i\cdot)_{-1/2}.$$

Again we see that the orthogonal complement to Σ_0 in this space coincides with $L^2(D,E_i)$-orthogonal complement to Σ_0. Denote π_{τ_Γ} the orthogonal projection on the subspace Σ_{τ_Γ}, consisting of sections with vanishing tangential parts on Γ. Definition 2.1 guarantees that the subspace Σ_{τ_Γ} is closed in $H^0_{A_i}(D,E_i)$. As $\tau_{i,\Gamma}(u(f)) = u^0 = 0$ then for all $v \in C^\infty_{comp}(D\cup\Gamma,E_{i-1})$ we obtain:

$$(u(f), A_{i-1}v)_D = (\pi_{\tau_\Gamma}u(f), A_{i-1}v)_{0,\tau_i} = (u(f), \pi_{\tau_\Gamma}A_{i-1}v)_{0,\tau_i} = (u(f), \pi_{\tau_\Gamma}A_{i-1}v)_D. \tag{31}$$

On the other hand, for all $g \in C^\infty_{comp}(D,E_{i+1})$ we have:

$$(\pi_{\tau_\Gamma}A_{i-1}v, A^*_i g)_D = (\pi_{\tau_\Gamma}A_{i-1}v, A^*_i g)_{0,\tau_i} = (A_{i-1}v, \pi_{\tau_\Gamma}A^*_i g)_{0,\tau_i} = (A_{i-1}v, A^*_i g)_D = 0,$$

because $A_i \circ A_{i-1} \equiv 0$. Therefore $A_i\pi_{\tau_\Gamma}A_{i-1}v = 0$ in D, and $\pi_{\tau_\Gamma}A_{i-1}v \in \Sigma_0$ for all $v \in C^\infty_{comp}(D\cup\Gamma,E_{i-1})$. Hence, formulae (30) and (31) and the fact that $u(f)$ is orthogonal to Σ_0 in $L^2(D,E_i)$, imply that $\nu_{i,\Gamma}(u(f)) = 0$ on Γ. □

Corollary 5.1. *Let $f \in H^{-1}(D,E_{i+1})$, $u^0 \in H^{-1/2}(\overline{\Gamma},E_i)$. If the Cauchy Problem 4.1 is solvable in the space $H^0_{A_i}(D,E_i)$ then the section $u(f,\tilde u_0) = u(f - A_i\mathcal{P}^{(D)}_{\Delta_i}\tau_i(\tilde u^0)) + \mathcal{P}^{(D)}_{\Delta_i}\tau_i(\tilde u^0)$ is also its solution satisfying $\nu_i(u(f,\tilde u_0)) = 0$ on Γ, $A^*_i u(f,\tilde u_0) = A^*_i\mathcal{P}^{(D)}_{\Delta_i}\tau_i(\tilde u^0)$ in D. Besides, if $f \in H^s_{loc}(D\cup\Gamma,E_{i+1})$, $u^0 \in H^{s+1/2}_{loc}(\Gamma,E_i)$ then $u(f,\tilde u_0) \in H^{s+1}_{loc}(D\cup\Gamma,E_i)$, $s \in \mathbb{Z}_+$.*

Proof. Let $u \in H^0_{A_i}(D,E_i)$ be a solution to Problem 4.1 with data $f \in H^{-1}(D,E_{i+1})$, $u^0 \in H^{-1/2}(\overline{\Gamma},E_i)$. Then, according to Lemma 2.1 and Remark 2.1, we have on Γ:

$$\tau_i(\tilde u^0) = \tau_i(u^0), \quad \tau_i(\mathcal{P}^{(D)}_{\Delta_i}\tau_i(\tilde u^0)) = \tau_i(u^0), \quad \nu_i(\mathcal{P}^{(D)}_{\Delta_i}\tau_i(\tilde u^0)) = 0.$$

Hence Problem 4.1 with data $\hat{f} = f - A_i \mathcal{P}^{(D)}_{\Delta_i} \tau_i(\tilde{u}^0) \in H^{-1}(D, E_{i+1})$ and $\hat{u}^0 = 0$ is solvable in the space $H^0_{A_i}(D, E_i)$; the section $\hat{u} = u - \mathcal{P}^{(D)}_{\Delta_i} \tau_i(\tilde{u}^0)$ is its solution. Therefore Theorem 5.1 implies that the section $u(f, \tilde{u}^0)$ is a solution to Problem 4.1 with data $f \in H^{-1}(D, E_{i+1})$, $u^0 \in H^{-1/2}(\overline{\Gamma}, E_i)$. By the construction it satisfies $\nu_i(u(f, \tilde{u}_0)) = 0$ on Γ, $A_i^* v(f, \tilde{u}_0) = A_i^* \mathcal{P}^{(D)}_{\Delta_i} \tau_i(\tilde{u}^0)$ in D.

Finally, if $f \in H^s_{loc}(D \cup \Gamma, E_i)$ then, using Theorem 5.1 and Lemma 2.1, we conclude that $t(u(f)) = 0$ on Γ,

$$\Delta_i u(f) = (A_i^* A_i + A_{i-1} A_{i-1}^*) u(f) = A_i^* f \in H^{s-1}_{loc}(D \cup \Gamma, E_i).$$

Therefore $u(f) \in H^{s+1}_{loc}(D \cup \Gamma, E_i)$, $s \in \mathbb{Z}_+$, because of Theorem on local improvement of smoothness for solutions to Dirichlet Problem (see, for instance, [2, Theorem 9.3.17]). Similarly, if $u^0 \in H^{s+1/2}_{loc}(\Gamma, E_i)$ then $\mathcal{P}^{(D)}_{\Delta_i} \tau_i(\tilde{u}^0) \in H^{s+1}_{loc}(D \cup \Gamma, E_i)$ according to Remark 2.1 and [2, Theorem 9.3.17]). Thus, $u(f, \tilde{u}_0)$ belongs to $H^{s+1}_{loc}(D \cup \Gamma, E_i)$, which was to be proved. $\square$

Since Corollary 5.1 practically reduces the Cauchy Problem 4.1 to the case with zero boundary data, we consider the situation in detail.

Theorem 5.2. *Let Δ_{i-1}, Δ_i, Δ_{i+1} satisfy the Uniqueness Condition* 1.1. *If $f \in H^{-1}(D, E_{i+1})$, $u^0 = 0$ then Problem* 4.1 *is solvable in the space $H^0_{A_i}(D, E_i)$ if and only if $A_{i+1} f = 0$ in D, $\tau_{i+1}(f) = 0$ on Γ and there is a section $\mathcal{F}_i \in L^2(\Omega, E_i) \cap S_{\Delta_i}(\Omega)$ coinciding with $T_i f$ in D^+.*

Proof. As $u^0 = 0$, then $F_i = T_i f$. Moreover, by Lemma 4.1, condition (19) is equivalent to the following two conditions: $A_{i+1} f = 0$ in D and $\tau_{i+1}(f) = 0$ on Γ. Now if there is a section $\mathcal{F}_i \in L^2(\Omega, E_i) \cap S_{\Delta_i}(\Omega)$ coinciding with $T_i f$ in D^+ then $(T_i f)^{\pm}, \mathcal{F}_i^{\pm} \in L^2(D^{\pm}, E_i) \cap S_{\Delta_i}(\Omega)$, $(T_i f - \mathcal{F}_i) \in \chi_D(L^2(D, E_i))$ and $A_i \Delta_i \mathcal{F}_i = 0$ in Ω. Therefore, it follows from Corollary 4.1 that Problem 4.1 is solvable in the space $H^0_{A_i}(D, E_i)$ if $A_{i+1} f = 0$ in D and $\tau_{i+1}(f) = 0$ on Γ. We note that formulae (22) and (24) yield:

$$u(f) = (T_i f - \mathcal{F}_i^-) \in L^2(D, E_i). \tag{32}$$

Back, if Problem 4.1 is solvable in the space $H^0_{A_i}(D, E_i)$ then $A_{i+1} f = 0$ in D, $\tau_{i+1}(f) = 0$ on Γ. Moreover, the extension $\mathcal{F}_{i,u} \in L^2(D, E_i) \cap S_{A_i \Delta_i}(\Omega)$ of the section $T_i f$ from D^+ on Ω is given by formula (22). Putting the solution $u(f)$ into (22) and using formula (23) and Definition 5.1, we obtain for all $v \in C^\infty_{comp}(\Omega, E_i)$:

$$-\langle \Delta_i \mathcal{F}_{i,u(f)}, v \rangle_\Omega = \langle \chi_D u(f), A_{i-1} A_{i-1}^* v \rangle_\Omega = (u(f), A_{i-1} A_{i-1}^* v)_D = (A_i^* u(f), A_i^* v)_D = 0,$$

because $\nu_{i,\Gamma}(u(f)) = 0$, $A_{i-1}^* u(f) = 0$ in D, i.e. $\Delta_i \mathcal{F}_{i,u(f)} = 0$ in Ω. $\square$

Remark 5.1. *Theorem 5.2 easily implies conditions of local solvability of the Cauchy problem for complex $\{A_i\}$ in $L^2(D, E_i)$ for $u^0 = 0$. Indeed, fix a point $x_0 \in \Gamma$. Let U be a (one-sided) neighborhood of x_0 in D and $\hat{\Gamma} = \partial U \cap \Gamma$. Set $\hat{F}_i = T_i \chi_U f$. As $F_i = \hat{F}_i + T_i \chi_{D \setminus U} f$ we see that F_i^+ extends as a solution to the Laplacian Δ_i in $\hat{\Omega} = U \cup \hat{\Gamma} \cup D^+$ if and only if the potential $\hat{F}_i^+$ does. Hence, under condition* (19), *the solution of the Cauchy problem exists in the neighborhood U where the extension of the potential F_i^+ does.*

Also we would like to note that Theorem 5.2 gives not only the solvability conditions to Problem 4.1 but the solution itself, of course, if it exists (see (32)). It is clear that we can use the theory of functional series (Taylor series, Laurent series, etc.) in order to get information about extendability of the potential $T_i^+ f$ (cf. [8], [2]). However in this paper we will use the theory of Fourier series with respect to the bases with the double orthogonality property (cf. [32], [2] or elsewhere). Moreover, using formula (32) we can construct approximate solutions of problem 4.1 in the Lebesgue space $L^2(D, E_i)$.

Lemma 5.1. *If $\omega \Subset \Omega$ is a domain with a piece-wise smooth boundary and $\Omega \setminus \omega$ has no compact (connected) components then there exists an orthonormal basis $\{b_\nu\}_{\nu=1}^\infty$ in $L^2(\Omega, E_i) \cap S_{\Delta_i}(\Omega)$ such that $\{b_{\nu|\omega}\}_{\nu=1}^\infty$ is an orthogonal basis in $L^2(\omega, E_i) \cap S_{\Delta_i}(\omega)$.*

Proof. In fact, these $\{b_\nu\}_{\nu=1}^\infty$ are eigen-functions of compact self-adjoint linear operator $R(\Omega, \omega)^* R(\Omega, \omega)$, where

$$R(\Omega, \omega) : L^2(\Omega, E_i) \cap S_{\Delta_i}(\Omega) \to L^2(\omega, E_i) \cap S_{\Delta_i}(\omega)$$

is the natural inclusion operator (see [2] or [9, theorem 3.1]). □

Now we can use the basis $\{b_\nu\}$ in order to simplify Theorem 5.2. For this purpose fix domains $\omega \Subset D^+$ and Ω as in Lemma 5.1 and denote by

$$c_\nu(T_i f^+) = \frac{(T_i f^+, b_\nu)_{L^2(\omega, E_i)}}{\|b_\nu\|^2_{L^2(\omega, E_i)}}, \quad \nu \in \mathbb{N},$$

the Fourier coefficients of $T_i f^+$ with respect to the orthogonal system $\{b_{\nu|\omega}\}$ in $L^2(\omega, E_i)$.

Corollary 5.2. *Let $f \in H^{-1}(D, E_{i+1})$, $u^0 = 0$. Problem 4.1 is solvable in the space $H^0_{A_i}(D, E_i)$ if and only if $A_{i+1} f = 0$ in D, $\tau_{i+1}(f) = 0$ on Γ and and the series $\sum_{\nu=1}^\infty |c_\nu(T_i f^+)|^2$ converges.*

Proof. Indeed, if Problem 4.1 is solvable in $L^2(D, E_i)$ then, according to Theorem 5.2 condition (19) is fulfilled, and there exists a function $\mathcal{F}_i \in L^2(\Omega, E_i) \cap S_{\Delta_i}(\Omega)$ coinciding with $T_i f^+$ in ω. By Lemma 5.1 we conclude that

$$\mathcal{F}_i(x) = \sum_{\nu=1}^\infty k_\nu(\mathcal{F}_i) b_\nu(x), \quad x \in \Omega, \tag{33}$$

where $k_\nu(\mathcal{F}_i) = (\mathcal{F}_i, b_\nu)_{L^2(\Omega, E_i)}$, $\nu \in \mathbb{N}$, are the Fourier coefficients of $\mathcal{F}_i$ with respect to the orthonormal basis $\{b_\nu\}$ in $L^2(\Omega, E_i) \cap S_{\Delta_i}(\Omega)$. Now Bessel's inequality implies that the series $\sum_{\nu=1}^\infty |k_\nu(\mathcal{F}_i)|^2$ converges.

Finally, the necessity of the corollary holds true because

$$c_\nu(T_i f^+) = \frac{(R(\Omega, \omega)\mathcal{F}_i, R(\Omega, \omega) b_\nu)_{L^2(\omega, E_i)}}{(R(\Omega, \omega) b_\nu, R(\Omega, \omega) b_\nu)_{L^2(\omega, E_i)}} = \frac{(\mathcal{F}_i, R(\Omega, \omega)^* R(\Omega, \omega) b_\nu)_{L^2(\Omega, E_i)}}{(b_\nu, R(\Omega, \Omega)^* R(\Omega, \omega) b_\nu)_{L^2(\omega, E_i)}} = k_\nu(\mathcal{F}_i).$$

Back, if the hypothesis of the corollary holds true then we invoke the Riesz-Fisher theorem. According to it, in the space $L^2(\Omega, E_i) \cap S_{\Delta_i}(\Omega)$ there is a section

$$\mathcal{F}_i(x) = \sum_{\nu=1}^\infty c_\nu(T_i f^+) b_\nu(x), \qquad x \in \Omega. \tag{34}$$

By the construction, it coincides with $T_i f^+$ in ω. Therefore, using Theorem 5.2, we conclude that Problem 4.1 is solvable in $L^2(D, E_i)$. $\square$

The examples of bases with the double orthogonality property be found in [9], [2], [32].

Let us obtain Carleman's formula for the solution of Problem 4.1. For this purpose we introduce the following Carleman's kernels:

$$\mathfrak{C}_N(y,x) = (A_i^*)'_y \Phi_i(y,x) - \sum_{\nu=1}^{N} c_\nu((A_i^*)'_y \Phi_i(y,\cdot)) b_\nu(x),\ N \in \mathbb{N},\ x \in \Omega,\ y \notin \overline{\omega}, x \neq y.$$

Corollary 5.3. *If Problem 4.1 is solvable in $L^2(D, E_i)$ for data $u_0 = 0$ and $f \in L^2(D, E_{i+1}) \cap H^s_{loc}(D \cup \Gamma, E_{i+1})$ then $u(f)$ belongs to $H^{s+1}_{loc}(D \cup \Gamma, E_i)$ and the following Carleman formula holds:*

$$u(f)(x) = \lim_{N\to\infty} \int_D \langle \mathfrak{C}_N(\cdot, x), f\rangle_y \, dy \tag{35}$$

where the limit converges in the spaces $H^0_{A_i}(D, E_i)$ and $H^{s+1}_{loc}(D \cup \Gamma, E_i)$.

Proof. Since $\overline{\omega} \cap \overline{D} = \emptyset$, using Fubini Theorem we have for all $\nu \in \mathbb{N}$:

$$c_\nu(T_i f^+) = \int_D \langle c_\nu((A_i^*)'_y \Phi_i(y,\cdot)), f\rangle_y \, dy.$$

This exactly yields identity (35) after applying Corollary 5.2, formula (34) and regrouping the summands in (32).

Besides, since $\mathcal{F}_i$ and each function b_ν are solutions of the elliptic system Δ_i in Ω, the Stiltjes-Vitali theorem implies that the series (34) converges in $C^\infty_{loc}(\Omega, E_i)$. Therefore we additionally conclude that the limit converges to $u(f)$ in $H^{s+1}_{loc}(D\cup\Gamma, E_i)$ because $T_i f \in H^1(D, E_i) \cap H^{s+1}_{loc}(D \cup \Gamma, E_i)$ due to the transmission property (see [31]). $\square$

Considering general complexes with smooth coefficients we arrive to the following natural question: under what conditions on the domain D the complex $\{A_i\}$ is exact at the positive degrees ? As far as we know there is no answer in the general situation. It is known that the formally exact differential elliptic complexes with real analytic coefficients are locally exact at the positive degrees (see, for instance, [15], [14]). Of course, all the Hilbert complexes with constant coefficients are exact at the positive degrees over the spaces of distrubutions in convex domains (see, for instance, [33]. Thus we are to consider this most investigated situation. However we emphasize that the use of the above proposed approach to the Cauchy problem for the elliptic complexes does not involve the information on the exactness of the complex!

6 Complexes with constant coefficients

Now we are to discuss examples for complexes with constant coefficients. Actually we can say much more, at least for domains of the special type.

Corollary 6.1. *Let (3) be an elliptic first order complex with constant coefficients in $\mathbb{R}^n$. If $\partial D \setminus \Gamma$ is a part of a strictly convex domain $\Omega \supset D$, then for any section $w \in C^\infty(\overline{D}, E_i)$ there* *is a section $h \in L^2(D, E_{i-1}) \cap C^\infty_{loc}(D \cup \Gamma, E_{i-1})$ such that $\tau_i(A_{i-1}h) = 0$ on Γ and the following formula holds true:*

$$w(x) = \mathcal{P}^{(D)}_{\Delta_i} \chi_\Gamma \tau_i(w))x) + \lim_{N\to\infty} \int_D \langle \mathfrak{C}_N(\cdot, x), A_i(u - \mathcal{P}^{(D)}_{\Delta_i} \chi_\Gamma \tau_i(w))\rangle_y \, dy + A_{i-1}h(x), \qquad (36)$$

where the limit converges in the spaces $H^0_{A_i}(D, E_i)$ and $C^\infty_{loc}(D \cup \Gamma, E_i)$.

Proof. Under the hypothesis of the corollary, Problem 4.1 is solvable for the data $\tau_{i,\Gamma}(u) \in C^\infty(\overline{\Gamma}, E_i)$ and $A_i u \in C^\infty(\overline{D}, E_{i+1})$. Extending $\tau_{i,\Gamma}(w)$ by zero onto all the boundary of D, we obtain $\tilde{w}_0 = \chi_\Gamma \tau_i(w) \in L^2(\partial D, E_i)$. Now Corollary 5.1 implies that the section $u(A_i w, \tilde{w}_0)$ belongs to the space $L^2(D, E_i) \cap C^\infty_{loc}(D \cup \Gamma, E_i)$ and $v = w - u(A_i w, \tilde{u}_0) \in \Sigma_0 \cap C^\infty_{loc}(D \cup \Gamma, E_i)$.

Denote v_0 the extension by zero of v from D on Ω. Clearly, $v_0 \in L^2(\Omega, E_i)$. As $\tau_{i,\Gamma}(v) = 0$, then $A_{i+1}v_0 = 0$ in Ω and hence there is a section $\tilde{h} \in L^2(\Omega, E_{i-1}) \cap H^1_{loc}(\Omega, E_{i-1})$ such that $A_{i-1}\tilde{h} = v_0$ in Ω (see, for instance, [33]). Set $h = \tilde{h} - A_{i-2}\Phi_{i-2}\chi_D A^*_{i-2}\tilde{h}$. Then

$$A^*_{i-2}\Delta_{i-1}h = A^*_{i-2}A_{i-2}A^*_{i-2}\tilde{h} - A^*_{i-2}A_{i-2}\chi_D A^*_{i-2}\tilde{h} = 0 \text{ in } D,$$

$$A_{i-1}\Delta_{i-1}h = A_{i-1}A^*_{i-1}A_{i-1}\tilde{h} = A_{i-1}A^*_{i-1}v \text{ in } D,$$

As the operator $(A_{i-1} \oplus A^*_{i-2})\Delta_{i-1}$ has injective symbol and

$$(A_{i-1} \oplus A^*_{i-2})\Delta_{i-1}h = (A_{i-1}A^*_{i-1}v, 0) \in C^\infty_{loc}(D \cup \Gamma, (E_i, E_{i-2})),$$

we see that $h \in C^\infty_{loc}(D \cup \Gamma, E_{i-1})$ satisfies $A_{i-1}h = w$ in D. Thus, $v = u(A_i w, \tilde{u}_0) + A_{i-1}h$ and formula (36) follows from Corollary 5.3. $\square$

At the conclusion let us consider two examples.

Example 6.1. Let (3) be the de Rham complex over $\mathbb{R}^n$, i.e E_i be the bundle of the exterior differential forms of the degree i and A_i be the differentiation operator d_i for the exterior differential forms. Choosing coordinates $x = (x_1, ..., x_n) \in \mathbb{R}^n$ we have for a form $u \in C^\infty(\mathbb{R}^n, \Lambda^i)$:

$$u = \sum_{|I|=i} u_I(x)\, dx_I, \quad d_i u = \sum_{j=1}^n \sum_{|I|=i} \frac{\partial u_I}{\partial x_j}(x)\, dx_j \wedge dx_I,$$

where $I = (j_1, \dots, j_i)$, $dx_I = dx_{j_1} \wedge \dots \wedge x_{j_i}$ and $\wedge$ is the exterior product for the differential forms.

Let $*$ be the Hodge operator for the differential forms (see, for instance, [15]), in particular, $dx_I \wedge *dx_I = dx$. Then $\Delta_i = \Delta I_{k(i)}$, where Δ is the usual Laplace operator in $\mathbb{R}^n$ and $I_{k(i)}$ is the unit $k(i) \times k(i)$-matrix. If $\Phi_i = I_{k(i)}\, \Phi$, where Φ is the standard fundamental solution to Δ of the convolution type, then M_i is the Norguet integral and (16) is the the Norguet integral formula (see, for instance, [15, §2.5]).

Let $\{h^{(j)}_\nu\}$ be the set of homogeneous harmonic polynomials forming a complete orthonormal system in the space $L^2(\partial B(0,1))$ on the unit sphere $\partial B(0,1)$ in $\mathbb{R}^n$, $n \geq 2$ (see [34, p. 453]). Therefore $\{h^{(j)}_{\nu|\partial B(0,1)}\}$ are *spherical harmonics* where ν is the homogeneity, j is the number of the polynomial of degree ν in the basis, $1 \leq j \leq J(\nu, n)$ with $J(\nu, n) = \frac{(n+2\nu-2)(n+\nu-3)!}{\nu!(n-2)!}$, $\nu > 0$,

$J(0,n) = 1$. It is easy to see that the system $\{h_\nu^{(j)}\}$ is orthogonal in $L^2(B(0,R))$ for any ball $B(0,R)$.

Let D be a part of the unit ball Ω cut off by a hypersurface $\Gamma \not\ni 0$. Then Carleman kernel in formulae (35), (36) has the following form:

$$\mathfrak{C}_N(y,x) = *_y d_y \Phi_i(y,x) - \sum_{|I|=i} \sum_{\mu=0}^{N} \sum_{j=1}^{J(\mu,n)} *_y d_y \left(\frac{h_\mu^{(j)}(y)\ dy_I}{|y|^{n+2\mu-2}(n+2\mu-2)} \right) h_\mu^{(j)}(x)\ dx_I.$$

We note that the operators d_i are non-zero for $0 \le i \le n-1$ only.

Hence for $n=1$ the operator d_0 is the usual differentiation and all the other operators d_i are identically zeros. Then the Cauchy problem for an interval $D = (a,b) \subset \mathbb{R}$ is well known: given a distribution f on (a,b) find a distribution u on $[a,b)$ such that

$$\begin{cases} u'(x) = f(x), & x \in (a,b), \\ u(a) = 0. \end{cases}$$

This problem is well-posed in the Sobolev spaces and its solution is given by the integral

$$u(x) = \int_a^x f(t)\ dt,$$

at least for f from the Sobolev spaces of a non-negative smoothness. For elements f from the Sobolev spaces of a negative smoothness the interpretations of the integral are also well known.

For $n=2$ the Cauchy problem for the de Rham complex at the degree $i=1$ can be inerpretated as follows. Let D be a bounded domain in $\mathbb{R}^2$ and

$$G = \{(x_1,x_2,x_3) : (x_1,x_2) \in D,\ 0 < x_3 < A\} \subset \mathbb{R}^3$$

be a cylinder with the base D. If we consder G as a bassin where the liquid behaves similarly in every section

$$D_b = \{(x_1,x_2,b) : (x_1,x_2) \in D,\ 0 < b < A\}$$

then the (stationary) flow of the ideal non-contractible liquid can be described by the system of equations

$$\begin{cases} \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} = h & \text{in} \quad D, \\ \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = g & \text{in} \quad D, \end{cases}$$

where the vector $u = (u_1,u_2)$ corresponds to the velocity vector of the fluid and the components h, g reflect the rotation points and the source points respectively (see, for example, [35, Ch. III, §2]). This exactly means

$$d_1 u = f \text{ in } D,\ d_0^* u = -g \text{ in } D,$$

for the differential forms

$$u(x) = u_1(x)dx_1 + u_2(x)dx_2, \quad f(x) = h(x)dx_1 \wedge dx_2, \quad g(x)$$

of the degrees 1, 2 and 0 respectively. If $(n_1(x), n_2(x))$ is the unit normal vector with respect to ∂D at the point x then

$$\tau_1(u) = n_2 u_1 - n_1 u_2 \text{ on } \partial D, \quad \nu_1(u) = n_1 u_1 + n_2 u_2 \text{ on } \partial D.$$

According to Theorem 5.2 the Cauchy problem for the de Rham complex in D with boundary data on $\Gamma \subset \partial D$, i.e.

$$\begin{cases} d_1 u = f & \text{in} \quad D, \\ \tau_1(u) = 0 & \text{on} \quad \Gamma, \end{cases}$$

is equivalent to the following problem

$$\begin{cases} d_1 v = f & \text{in} \quad D, \\ d_0^* v = 0 & \text{in} \quad D, \\ \tau_1(u) = 0 & \text{on} \quad \Gamma, \\ \nu_1(u) = 0 & \text{on} \quad \Gamma. \end{cases}$$

The last one is obviously the Cauchy problem for the classical Cauchy-Riemann system with respect to the function $w(z) = v_2(x_1, x_2) + \sqrt{-1} v_1(x_1, x_2)$ with $z = x_1 + \sqrt{-1} x_2$:

$$\begin{cases} \frac{\partial w}{\partial \bar{z}} = f/2 & \text{in} \quad D, \\ w = 0 & \text{on} \quad \Gamma, \end{cases}$$

where $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x_1} + \sqrt{-1}\frac{\partial}{\partial x_2})$. Thus, according to Hadamard's example (see [1]) the Cauchy problem for the de Rham complex in $\mathbb{R}^2$ at the degree 1 is ill-posed in all the standard functional spaces (the spaces of smooth functions, the Sobolev spaces etc.).

For $n = 3$ the operators d_0, d_1, d_2 can be identifyed with the famuos gradient operator ∇, the rotor operator rot and the divergence operator div respectively which are widely used in Mechanics, Hydrodinamics, Electrodynamics and so on:

$$d_0 \approx \nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix}, \quad d_1 \approx \mathrm{rot} = \begin{pmatrix} 0 & -\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} & 0 & -\frac{\partial}{\partial x_1} \\ -\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \end{pmatrix}, \quad d_2 \approx \mathrm{div} = \begin{pmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \end{pmatrix},$$

$$d_0^* \approx -\mathrm{div}, \quad d_2^* \approx \mathrm{rot}, \quad d_2^* \approx -\nabla$$

For instance, according to Theorem 5.2, the Cauchy problem for the de Rham complex at the degree 1 for a domain $D \subset \mathbb{R}^n$, a set $\Gamma \subset \partial D$ and a datum $f = (f_1, f_2, f_3)$ is equivalent to the Cauchy problem for the (stationary) Maxwell type system with respect to the vector function $u = (u_1, u_2, u_3)$:

$$\begin{cases} \mathrm{rot}\, u = f & \text{in} \quad D, \\ \mathrm{div}\, u = 0 & \text{in} \quad D, \\ u = 0 & \text{on} \quad \Gamma. \end{cases}$$

We refer to [36] for applications of the theory of differential complexes to the investigation of the Maxwell type equations.

Example 6.2. Let (3) be the Dolbeault complex over $\mathbb{C}^n$, i.e E_i be the bundle of exterior differential forms of bi-degree $(0,i)$ and A_i be the Cauchy-Riemann operator $\overline{\partial}_i$ for the exterior differential forms. Choosing coordinates $z=(z_1,...,z_n)$ with $z_j=x_j+\sqrt{-1}x_{j+n}$, $j=1,...,n$, and $x=(x_1,...,x_{2n})\in\mathbb{R}^{2n}$ we have for a form $u\in C^\infty(\mathbb{C}^n,\Lambda^{(0,i)})$:

$$u=\sum_{|I|=i}u_I(z)\ d\overline{z}_I,\quad \overline{\partial}_iu=\sum_{j=1}^n\sum_{|I|=i}\frac{\partial u_I}{\partial\overline{z}_j}(z)\ d\overline{z}_j\wedge d\overline{z}_I,$$

where $\frac{\partial}{\partial\overline{z}_j}=\frac{1}{2}\left(\frac{\partial}{\partial x_j}+\sqrt{-1}\frac{\partial}{\partial x_{j+n}}\right)$, $dz_j=dx_j+\sqrt{-1}dx_{j+n}$, $I=(j_1,\dots,j_i)$, $d\overline{z}_I=d\overline{z}_{j_1}\wedge\cdots\wedge d\overline{z}_{j_i}$.

It is well known that $\star u=\overline{*u}$ for a form u with $*$ being the Hodge operator for the differential forms (see [37, §14]). Then $\Delta_i=1/2\ \Delta I_{k(i)}$, where Δ is the usual Laplace operator in $\mathbb{R}^{2n}$ and $I_{k(i)}$ is the unit $k(i)\times k(i)$-matrix. If $\Phi_i=I_{k(i)}\ \Phi$, where Φ is the standard fundamental solution to Δ of the convolution type, then M_i is the Martinelli-Bochner-Koppelmann integral and (16) is the the Martinelli-Bochner-Koppelmann integral formula (see, for instance, [38] or [15]).

Let D be a part of the unit ball Ω cut off by a hypersurface $\Gamma\not\ni 0$. Then Carleman kernel in formulae (35), (36) has the following form (see [21]):

$$\mathfrak{C}_N(\zeta,z)=\star_\zeta\overline{\partial}_\zeta\Phi_i(\zeta,z)-\sum_{|I|=i}\sum_{\mu=0}^{N}\sum_{j=1}^{J(\mu,2n)}\star_\zeta\overline{\partial}_\zeta\left(\frac{\overline{h^{(j)}_\mu(\zeta)}\ d\overline{\zeta}_I}{|\zeta|^{2n+2\mu-2}(2n+2\mu-2)}\right)\ h^{(j)}_\mu(z)\ d\overline{z}_I$$

where $\{h^{(j)}_\mu\}$ is the system of the spherical harmonics (see Example 6.1).

A result similar to Corollary 6.1 was obtained in [18, Theorem 3.1] for the Dolbeault complex if $\partial D\setminus\Gamma$ is i-strictly pseudo concave hypersurface; however they had no aim to prove that the tangential part of the rest $\overline{\partial}_ih$ vanished on Γ.

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